New results in the reverse mathematics analysis of Ramsey theory

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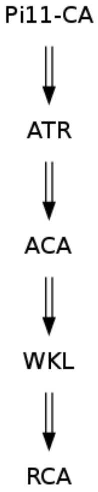
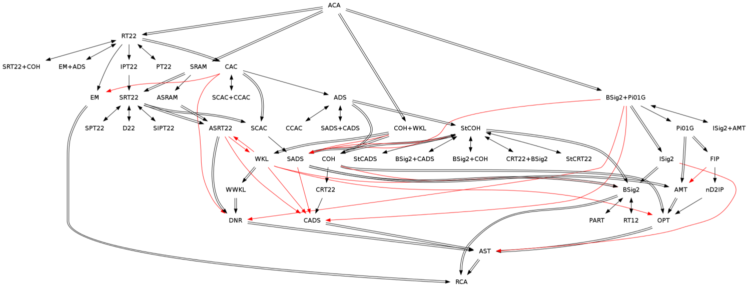
Summary

Plan

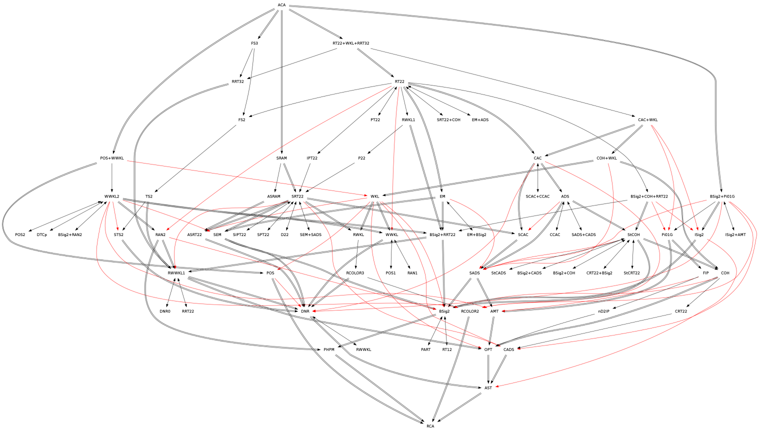
The "big five" subsystems



Previous reverse mathematics zoo



Current reverse mathematics zoo



Plan

Martin-Löf Randoms & D.N.C functions

Definition (Martin-Löf Random)

An infinite sequence S is Martin-Löf random if and only if

$$(\exists c)(\forall \sigma \prec S)(K(\sigma) \geq |\sigma| - c)$$

Definition (Diagonally Non-Computable function)

A function f is DNC if

$$(\forall e)(f(e) \neq \Phi_e(e))$$

Martin-Löf Randoms vs D.N.C functions

Theorem (Kjos-Hanssen, Miller)

The following are equivalent.

- *A computes a DNC function.*
- *A computes an infinite subset of a 1-random set.*

Plan

Trees

Definition (Tree)

A tree T is a set closed under prefixes:

$$\forall \sigma \in T, \tau \prec \sigma \Rightarrow \tau \in T$$

Definition (Measure of a tree)

$$\mu(T) \stackrel{def}{=} \lim_{n \to \infty} \frac{\texttt{card}\{\sigma \in T \ : \ |\sigma| = n\}}{2^n}$$

Path & Homogeneous set

Definition (Path in a tree)

A set P is a *path* in a tree T if $(\forall \sigma \prec P)(\sigma \in T)$

Definition (Set homogeneous for a path in a tree)

A set H is *homogeneous for a path trough a tree T with color c* if

$$(\forall n)(\exists \sigma \in T)(\forall x)(x \in H \restriction n \rightarrow \sigma(x) = c)$$

König's Lemma & Ramsey-Type König's Lemma

Definition (Weak Weak König's Lemma)

Every subtree of $2^{<\omega}$ of positive measure has a path.

Definition (Ramsey-Type Weak Weak König's Lemma)

Every subtree of $2^{<\omega}$ of positive measure has a set homogeneous for a path.

Some results

Theorem (A. Kucera, 1985)

A Martin-Löf random is a path (up to prefix) in a tree iff the tree has positive measure.

Theorem

There is a tree of positive capturing only Martin-Löf randoms.

Over $\mathbf{RCA_0}$...

Does this still hold over $\mathbf{RCA_0}$?

Over $\mathbf{RCA_0}$...

RAND (Martin Lof Random)

$\mathbf{RCA_0}$ + "For every X there is a random relative to X".

DNC (Diagonally Non-Computable)

$\mathbf{RCA_0}$ + "For every X there is a function DNC relative to X".

WWKL (Weak Weak König's Lemma)

$\mathbf{RCA_0}$ + "Every binary tree **of positive measure** has a path".

RWWKL (Ramsey-Type Weak Weak König's Lemma)

$\mathbf{RCA_0}$ + "Every binary tree T **of positive measure** has an infinite set homogeneous for a path in T".

Over $\mathbf{RCA_0}$...

Theorem (Avigad, Dean, & Rute)

$\mathbf{RCA_0} \vdash \mathbf{RAND} \leftrightarrow \mathbf{WWKL_0}$

Theorem (Giusto, Simpson)

$\mathbf{RCA_0} \vdash \mathbf{WWKL_0} \rightarrow \mathbf{DNC}$

Theorem (Ambos-Spies, Kjos-Hanssen, Lempp & Slaman)

$\mathbf{RCA_0} \nvdash \mathbf{DNC} \rightarrow \mathbf{WWKL_0}$

Over $\mathbf{RCA_0}$...

Theorem (Flood)

$\mathbf{RCA_0} \vdash \mathbf{WWKL_0} \rightarrow \mathbf{RWWKL_0}$

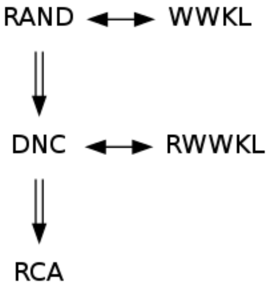
Theorem (Flood)

$\mathbf{RCA_0} \vdash \mathbf{RWWKL_0} \rightarrow \mathbf{DNC}$

Theorem (Bienvenu, Patey & Shafer)

$\mathbf{RCA_0} \vdash \mathbf{DNC} \rightarrow \mathbf{RWWKL_0}$

Over $\mathbf{RCA_0}$...



Plan

$0'$-randoms and $0'$-computable trees

Definition

A set S is $0'$-random if it is random relative to $0'$.

WWKL[$0'$]

Every subtree of $2^{<\omega}$ **of positive measure**, computable in $0'$ has a path.

RWWKL[$0'$]

Every subtree of $2^{<\omega}$ **of positive measure**, computable in $0'$ has a set homogeneous for a path.

$0'$-randoms and $0'$-computable trees

$\mathbf{RAND}[\mathbf{0'}]$ and $\mathbf{WWKL}[\mathbf{0'}]$ are almost equal (up to $\mathbf{B\Sigma_2}$).

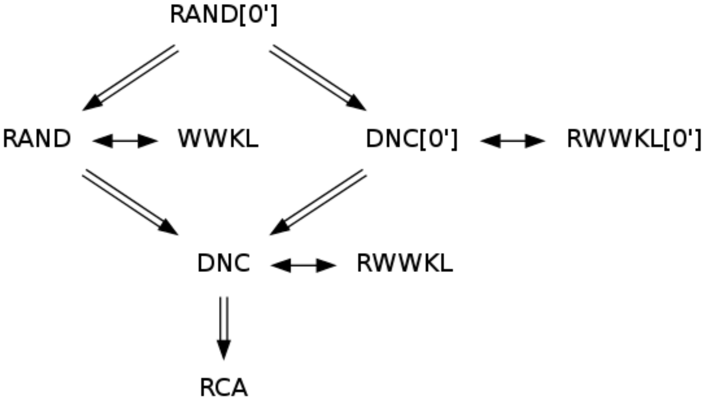
Theorem (Avigad, Dean, & Rute)

$\mathbf{RCA_0} \vdash \mathbf{RAND}[\mathbf{0'}] \to \mathbf{DNC}[\mathbf{0'}]$

Theorem (Bienvenu, Patey & Shafer)

$\mathbf{RCA_0} \vdash \mathbf{RWWKL}[\mathbf{0'}] \leftrightarrow \mathbf{DNC}[\mathbf{0'}]$

$0'$-randoms and $0'$-computable trees



Randomized algorithms

How powerful is randomness ?

Plan

Rainbow Ramsey Theorem

Definition (k-bounded function)

A coloring function $\mathbb{N}^n \to \mathbb{N}$ is k-bounded if $\mathtt{card}\,\{x \in \mathbb{N}^n : f(x) = c\} \leq k$ for every color c.

$\mathbf{RRT^n_k}$ (Rainbow Ramsey Theorem)

For every k-bounded coloring function $f : \mathbb{N}^n \to \mathbb{N}$ there is an infinite set H such that $f \restriction H^n$ is injective.

Rainbow Ramsey Theorem

Theorem (Miller)

$\mathbf{RCA_0} \vdash \mathbf{DNC}[\mathbf{0}'] \leftrightarrow \mathbf{RRT^2_2}$

Erdös-Moser Theorem

Definition (Tournament)

A tournament is a set $T \subseteq \mathbb{N} \times \mathbb{N}$ such that

$$(x, y) \in T \leftrightarrow (y, x) \notin T$$

Definition (Transitive tournament)

A tournament T is transitive if

$$(x, y) \in T \wedge (y, z) \in T \rightarrow (x, z) \in T$$

Definition (Stable tournament)

A tournament T is stable if

$$(\forall x)(\exists y)[(\forall z > y)((x, z) \in T) \vee (\forall z > y)((x, z) \notin T)]$$

Erdös-Moser Theorem

EM (Erdös-Moser Theorem)

$\mathbf{RCA_0}$ + "Every infinite tournament has an infinite transitive subtournament".

SEM (Stable Erdös-Moser Theorem)

$\mathbf{RCA_0}$ + "Every stable infinite tournament has an infinite transitive subtournament".

Erdös-Moser Theorem

Theorem (Bienvenu, Patey & Shafer)

The following statements are true over $\mathbf{RCA_0}$

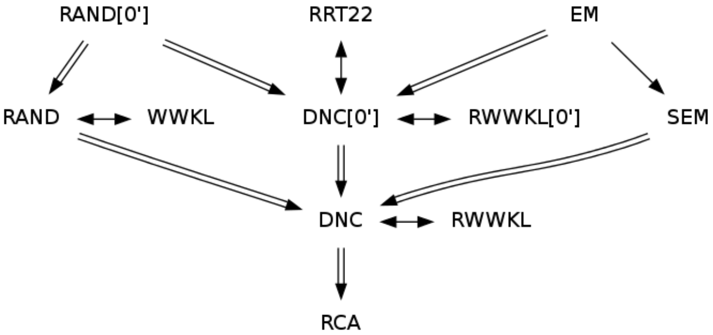
- $\mathbf{EM} \rightarrow \mathbf{DNC}[\mathbf{0}']$
- $\mathbf{SEM} \rightarrow \mathbf{DNC}$

Idea: *Diagonalize (modulo encoding) against finite* $0'$*-c.e. sets using tournaments (respectively finite c.e. sets using stable tournaments).*

Question

Is there a direct proof of $\mathbf{RCA_0} \vdash \mathbf{EM} \rightarrow \mathbf{RRT^2_2}$ *?*

Revised zoo



Revised zoo

Does **EM** imply $\mathbf{WWKL_0}$ over $\mathbf{RCA_0}$?

Ramsey Theorem

$\mathbf{RT^n_k}$ (Ramsey theorem)

$\mathbf{RCA_0}$ + "For every coloring function $f : \mathbb{N}^n \to \{0, .., k\}$ there is an infinite set H such that $f \restriction H^n$ is monochromatic."

Ramsey Theorem

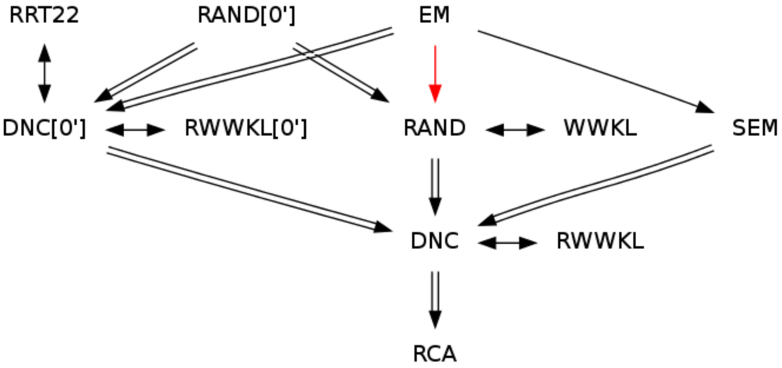
Theorem (Lerman, Solomon, Towsner)

$\mathbf{RCA_0} \vdash \mathbf{RT^2_2} \rightarrow \mathbf{EM}$

Theorem (Liu)

$\mathbf{RCA_0} \nvdash \mathbf{RT^2_2} \rightarrow \mathbf{WWKL_0}$

Revised zoo



Revised zoo

Does **SEM** imply $\mathbf{DNC}[\mathbf{0'}]$ over $\mathbf{RCA_0}$?

Stable Ramsey Theorem

Definition (Stable function)

A function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is stable if

$$(\forall x)(\exists y)(\forall z > y)(f(x, z) = f(x, y))$$

$\mathbf{SRT^n_k}$ (Stable Ramsey theorem)

$\mathbf{RCA_0}$ + "For every stable coloring function $f : \mathbb{N}^n \to \{0, .., k\}$ there is an infinite set H such that $f \upharpoonright H^n$ is monochromatic."

Stable Ramsey Theorem

Theorem (Lerman, Solomon, Towsner)

$\mathbf{RCA_0} \vdash \mathbf{SRT^2_2} \rightarrow \mathbf{SEM}$

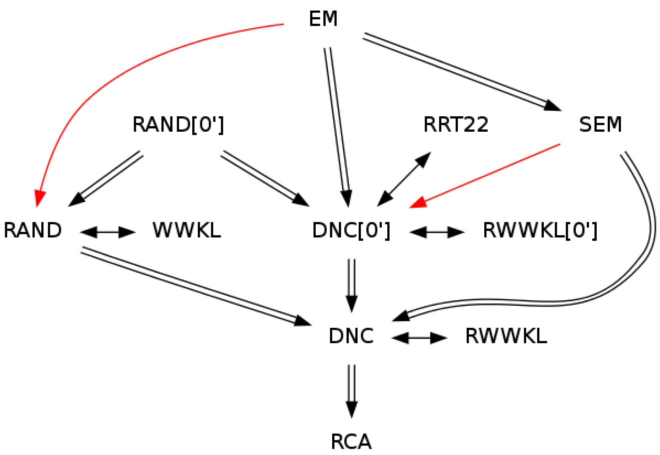
Theorem (Chong, Slaman, Yang)

There exists a non-standard model of $\mathbf{SRT^2_2}$ *with only low sets.*

Corollary

$\mathbf{RCA_0} \nvdash \mathbf{SRT^2_2} \rightarrow \mathbf{DNC}[\mathbf{0'}]$

Revised zoo



Revised zoo

Does $\mathbf{RAND}[\mathbf{0}']$ imply $\mathbf{SEM}$ over $\mathbf{RCA_0}$?

No Randomized Algorithm Property

Definition

A principle has the *NRA property* if adding randoms to the standard model (almsot surely) doesn't realize the principle.

Tip: A way to prove that a principle has the NRA property consist of creating an instance whose class of solutions is almost surely non-computed by an oracle.

No Randomized Algorithm Property

In particular, if a principle $\mathbf{P}$ has the NRA property, then

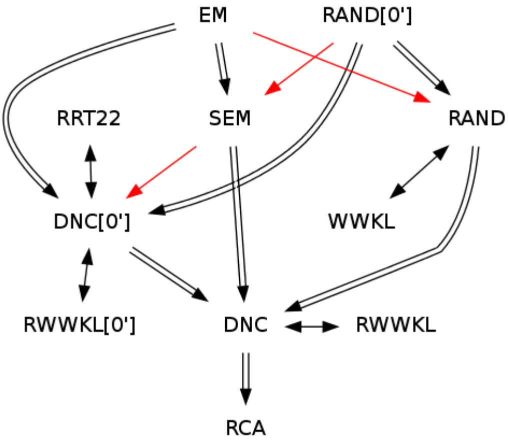
$$\mathbf{RCA_0} \nvdash \mathbf{RAND}[\mathbf{0}'] \to \mathbf{P}$$

Theorem (Bienvenu, Patey, Shafer)

SEM *has the NRA property*

Idea: *A "measure-risking argument" (like Antoine's talk) combined with a priority construction with finite injury.*

Revised zoo



Plan

Ramsey Graph coloring

Definition (Ramsey Graph coloring)

$\mathbf{RCOLOR_k}$: Every locally k-colorable graph has an infinite monochromatic set.

Theorem (Bienvenu, Patey, Shafer)

$\mathbf{RCOLOR_2}$ *has the NRA property*

Tip: *Still a "measure-risking argument" combined with a priority construction with finite injury + a combinatorial argument.*

Ramsey Graph coloring

Theorem (Bienvenu, Patey, Shafer)

$\mathbf{RCA_0} \vdash \mathbf{RCOLOR_3} \rightarrow \mathbf{DNC}$

Tip: *Tricky proof involving coding via widgets.*

Question

Does $\mathbf{RCOLOR_2}$ *imply* $\mathbf{DNC}$ *over* $\mathbf{RCA_0}$ *?*

Plan

References

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Questions

Thank you for listening !