Introduction to Reverse Mathematics

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January 29, 2014

INTRODUCTION

PROBLEM 1

Question (Bac S, 2006)

Prove Gauss theorem using Bézout theorem.

- How to express the relation between theorems ?
- What does it mean that a true theorem implies another true theorem ?

PROBLEM 2

Theorem
If $n \equiv 0 \mod 4$ *and* $n > 42$ *then* n *is even.*

- Are our theorems optimal ?
- Are all the premises truly required ?
- How to formalize "optimality" ?

THE ANSWER

Reverse Mathematics

PROBLEM 1 : A FIRST ATTEMPT

- How to express the relation between theorems ?
- What does it mean that a true theorem implies another true theorem ?

Example

$ZF + AC \vdash$ Zorn's lemma

- What about theorems provable in ZF ?

... use a weaker system

WHAT TO EXPECT OF SUCH A SYSTEM ?

- Many theorems should not be provable over it.
- Not too weak (eg. invariant under coding)
- Should give some insights on theorems.

WHAT DO WE FOCUS ON ?

- Constructivity
  - Type theory
  - Intuitionistic Reverse mathematics
- Time/Space constraints
  - Complexity theory
- Effectiveness
  - Computability
  - Reverse mathematics

WHAT IS EFFECTIVENESS ? (FOR US)

$$\text{Effectiveness} \neq \text{constructivity}$$

WHAT IS EFFECTIVENESS ? (STILL FOR US)

Theorem
For every infinite binary sequence there exists a value which appears infinitely many times.

- Effective theorem
- Not constructive

SETTLING REVERSE MATHEMATICS

To formalize Reverse Mathematics we need

- a language
- a logic
- a base theory

WHICH LANGUAGE ?

- Express naturally most ordinary theorems

- (König's lemma) *Every infinite finitary branching tree has an infinite path.*
- (Bolzano Weierstrass) *Every infinite sequence has an infinite converging subsequence.*
- (Ramey theorem) *Every coloring of tuples with finitely many colors has an infinite monochromatic subset.*
- ...

Which language ?

Remark

Most of our theorems are of the form

$$(\forall X)(\exists Y)\Phi(X, Y)$$

where Φ *has only first order quantifiers*

Example (König's lemma)

$$(\forall X)(\exists Y)[\mathit{Tree}(X) \rightarrow \mathit{Path}(Y, X)]$$

WHICH LANGUAGE ?

Second order arithmetic (L_2)

Numerical terms

$$t ::= 0 \mid 1 \mid x \mid t_1 + t_2 \mid t_1 \cdot t_2$$

Formulas

$$\begin{aligned} f ::= \quad & t_1 = t_2 \mid t_1 < t_2 \mid t_1 \in X \mid \forall x.f \\ & \mid \exists x.f \mid \forall X.f \mid \exists X.f \mid \neg f \mid f_1 \vee f_2 \end{aligned}$$

Which logic ?

- Intuitionistic ⇒ Intuitionistic reverse mathematics
- Classical ⇒ Reverse mathematics

Historical reasons ?

WHICH AXIOMS ?

- Effectiveness is about infinite objects
- Finite objects should behave as usual

Basic axioms

$$
\begin{array}{ll}
n+1 \neq 0 & m+1=n+1 \Rightarrow m=n \\
m+0=m & m+(n+1)=(m+n)+1 \\
m \cdot 0=0 & m \cdot (n+1)=(m \cdot n)+m \\
\neg m<0 & m<n+1 \Leftrightarrow (m<n \vee m=n)
\end{array}
$$

WHICH AXIOMS ?

- Full second order arithmetic (Z_2) ?

Induction axiom

$$(0 \in X \wedge \forall n.(n \in X \Rightarrow n+1 \in X)) \Rightarrow \forall n.(n \in X)$$

Comprehension scheme

$$\exists X.\forall n.(n \in X \Leftrightarrow \varphi(n))$$

where $\varphi(n)$ is any formula of L_2 in which X does not occur freely.

WHICH AXIOMS ?

- Most ordinary theorems are already provable in Z_2
- No notion of effectiveness in proofs in Z_2

EFFECTIVENESS

Definition (Σ^0_1, Π^0_1 and Δ^0_1 relations)

- Σ^0_1 : definable by a formula $\exists n.\phi$
- Π^0_1 : definable by a formula $\forall n.\phi$
- Δ^0_1 : both Σ^0_1 and Π^0_1

where ϕ is a L_2-formula containing only bounded quantifiers.

Theorem (Post's theorem)

A set A is computably enumerable (resp. computable) in $B_1, B_2, \ldots$ iff it is definable by a Σ^0_1 relation (resp. a Δ^0_1 relation) with parameters $B_1, B_2, \ldots$.

WHICH AXIOMS ?

Δ^0_1 Comprehension scheme

$$\forall n(\varphi(n) \Leftrightarrow \psi(n)) \Rightarrow \exists X.\forall n.(x \in X \Leftrightarrow \varphi(n))$$

where $\varphi(n)$ is any Σ^0_1 formula of L_2 in which X does not occur freely and $\psi(n)$ is any Π^0_1 formula of L_2.

Which axioms ?

Σ^0_1 Induction scheme

$$(\varphi(0) \wedge \forall n.(\varphi(n) \Rightarrow \varphi(n+1))) \Rightarrow \forall n.\varphi(n)$$

where $\varphi(n)$ is any Σ^0_1 formula of L_2

- Σ^0_1 can be replaced by Δ^0_1

WE GET... RCA_0

Recursive Comprehension Axiom (RCA_0)

- Basic Peano axioms
- Σ^0_1 induction scheme
- Δ^0_1 comprehension scheme

HOW TO THINK ABOUT RCA_0 ?

RCA_0 captures computable mathematics

Many of our principles are of the form $(\forall X)(\exists Y)\Phi(X, Y)$.

- X is called an *instance*
- Y is called a *solution*

If there is a computable instance with no computable solution then RCA_0 does not prove the principle.

How to think about RCA_0 ?

RCA_0 captures computable mathematics

We can create a model $\mathcal{M}_0$ of RCA_0 taking

- ω as the first order part
- $COMP = \{X \in 2^\omega : X \text{ is computable } \}$ as second order part

- If we have a computable instance X then $X \in COMP$.
- If X has no computable solution, X has no solution in $COMP$.

Problem 1 revisited

- Formal framework to reason about theorems
- Is it weak enough ?

We can use the question about computable instance with no computable solution.

PROBLEM 1 REVISITED

Theorem

There exists an infinite computable binary tree with no infinite computable path.

$$\mathcal{M}_0 \not\models \text{König's lemma}$$

- This is the case for many theorems.

PROBLEM 2 REVISITED

- Are our premises optimal ?
- What does “optimal” mean ?

- Prove the premise assuming the conclusion.
- This can be formalized over RCA_0.

Problem 2 revisited

Observation 1
Most ordinary theorems are provable from weak axioms.

Observation 2
Most ordinary theorems are *equivalent* to those axioms.

PLAN

THE BATTLE


1
Gödel
VS
König
1
60/72
inflicted damage
103/112
0/0
9
0/0
00:23
BLOCK

WEAK KÖNIG'S LEMMA

Definition (Tree)

A set $T \subseteq 2^{<\omega}$ is a tree iff it is closed under prefixes:

$$\forall \sigma \in T, \tau \prec \sigma \Rightarrow \tau \in T$$

Definition (Path)

$P \subseteq \omega$ is a path in a tree T iff all prefixes of P are in T.

$$\forall \sigma \prec P, \sigma \in T$$

WEAK KÖNIG'S LEMMA

Definition (WKL_0)

RCA_0 + "Every infinite subtree of $2^{<\omega}$ has a path."

- We saw that $\mathsf{RCA}_0 \nvdash \mathsf{WKL}_0$

GÖDEL'S COMPLETENESS THEOREM

Definition (Countable model)

A *countable model* is a function $M : T_M \cup S_M \to |M| \cup \{0, 1\}$ where

- $|M| \subseteq \omega$ is the *universe of M*
- T_M is the set of closed terms
- S_M is the set of sentences

over $L_M = L \cup \{\underline{m} : m \in |M|\}$ where M obeys the familiar clauses of Tarski's truth definition.

GÖDEL'S COMPACTNESS THEOREM

Definition (Gödel's completeness theorem)

Every countable consistent set X of sentences has a model, i.e. there exists a countable model M such that $(\forall i)(\varphi_i \in X \to M(i) = 1)$.

Definition (Gödel's compactness theorem)

If each finite subset of X has a model, then X has a model.

- Compactness follows from completeness

KÖNIG 1 - GÖDEL 0

Theorem
$\mathsf{RCA}_0 \vdash \mathsf{WKL}_0 \to$ *Gödel's completeness theorem*

Part 1/3.

- Given an consistent enumeration X of sentences
  - Define a set of constant symbols $C = \langle \underline{c}_n : n \in \omega \rangle$.
  - Let Φ be the set of all formulas $\varphi(x)$ in $L_1 = L \cup C$.
  - Define $X_1 = X \cup \{(\exists x \varphi_n(x)) \to \varphi_n(\underline{c}_n) : n \in \omega\}$
- X_1 is consistent
- Fix an enumeration of all sentences $S_0, S_1, \ldots$ over L_1

□

KÖNIG 1 - GÖDEL 0

Theorem

$\mathsf{RCA}_0 \vdash \mathsf{WKL}_0 \to$ *Gödel's completeness theorem*

Part 2/3.

- Define a binary tree T as follows:
  - Add by default all nodes $\sigma \in 2^{<\omega}$
  - If at stage s, $X_1[s] \vdash S_i$, remove all σ such that $\sigma(i) = 0$.
  - If at stage s, $X_1[s] \vdash \neg S_i$, remove all σ such that $\sigma(i) = 1$.
- Any path X_1^* in T is a completion of X_1

$\square$

KÖNIG 1 - GÖDEL 0

Theorem
$\mathsf{RCA}_0 \vdash \mathsf{WKL}_0 \to$ *Gödel's completeness theorem*

Part 3/3.

- Build a model M of X_1 as follows:
  - Let $|M|$ be the set of $\underline{c}_n \in C$ such that $\neg\exists m(m < n \wedge (\underline{c}_m = \underline{c}_n) \in X_1^*$
  - For all $\varphi \in S_M$ set $M(\varphi) = 1$ iff $\varphi \in X_1^*$.

$\square$

KÖNIG 1 - GÖDEL 1

Theorem

$\mathsf{RCA}_0 \vdash$ *Gödel's compactness theorem* $\rightarrow \mathsf{WKL}_0$

Proof.

- Prop logic with atomic formulas $\langle a_n : n \in \omega \rangle$
- Given an infinite tree $T \subseteq 2^{<\omega}$
  - Define the formula (where $a_i^1 = a_i$ and $a_i^0 = \neg a_i$)

$$\varphi_n = \bigvee \left\{ \bigwedge \left\{ a_i^{\sigma(i)} : i < n \right\} : \sigma \in T, |\sigma| = n \right\}$$

  - φ_i is satisfiable
  - $\varphi_{n+1} \rightarrow \varphi_n$ is a tautology
  - $\{\varphi_n : n \in \omega\}$ is satisfiable
  - Any model of $\{\varphi_n : n \in \omega\}$ is a path in T

□

KÖNIG 1 - GÖDEL 1

Theorem
The following are equivalent over RCA_0*:*

- WKL_0
- *Gödel's completeness theorem*
- *Gödel's compactness theorem*

PLAN

A PERFECT WORLD ?

- Most of our theorems lie nicely in a few main subsystems.
- They are even equivalent to those systems.

- ... however a class of theorems breaks the rules

Ramseyan theorems

Theorem

Every coloring of tuples with two colors has an infinite monochromatic subset...mouahahah (evil laugh)

References

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Questions

Thank you for listening !