Reverse mathematics: Classifying principles by the no randomized algorithm property.

Laurent Bienvenu Ludovic Patey Paul Shafer

LIAFA, Université Paris 7

September 23, 2013

Summary

Plan

What is Reverse Mathematics ?

Definition

Program in mathematical logic that seeks to determine which axioms are required to prove theorems of mathematics.

- Weak system (RCA_0)
- Prove equivalence of theorems and axioms over RCA_0
- Lattice of systems

Applications

- Soundness
- Heuristic for new proofs

Reverse Mathematics

Observation

Most theorems of "ordinary" mathematics

- live in weak systems.
- are equivalent to axioms over RCA_0

- Refine our structure of weak systems.
- Weaker than Ramsey theorem and König's lemma.

Language of Second Order Arithmetic L_2

Numerical terms

$$t ::= 0 \mid 1 \mid x \mid t_1 + t_2 \mid t_1 \cdot t_2$$

Formulas

$$\begin{aligned} f ::= \quad & t_1 = t_2 \mid t_1 < t_2 \mid t_1 \in X \mid \forall x.f \\ & \mid \exists x.f \mid \forall X.f \mid \exists X.f \mid \neg f \mid f_1 \vee f_2 \end{aligned}$$

Axioms of Second Order Arithmetic Z_2

Basic axioms

$$
\begin{array}{ll}
n+1 \neq 0 & m+1 = n+1 \Rightarrow m = n \\
m+0 = m & m+(n+1) = (m+n)+1 \\
m \cdot 0 = 0 & m \cdot (n+1) = (m \cdot n)+m \\
\neg m < 0 & m < n+1 \Leftrightarrow (m < n \vee m = n)
\end{array}
$$

Induction axiom

$$(0 \in X \wedge \forall n.(n \in X \Rightarrow n+1 \in X)) \Rightarrow \forall n.(n \in X)$$

Comprehension scheme

$$\exists X.\forall n.(n \in X \Leftrightarrow \varphi(n))$$

where $\varphi(n)$ is any formula of L_2 in which X does not occur freely.

Subsystem of Z_2

Definition (Subsystem of Z_2)

System based of L_2 whose axioms are theorems of Z_2

The system RCA_0

Basic axioms

Σ^0_1 Induction scheme

$$(\varphi(0) \wedge \forall n.(\varphi(n) \Rightarrow \varphi(n+1))) \Rightarrow \forall n.\varphi(n)$$

where $\varphi(n)$ is any Σ^0_1 formula of L_2

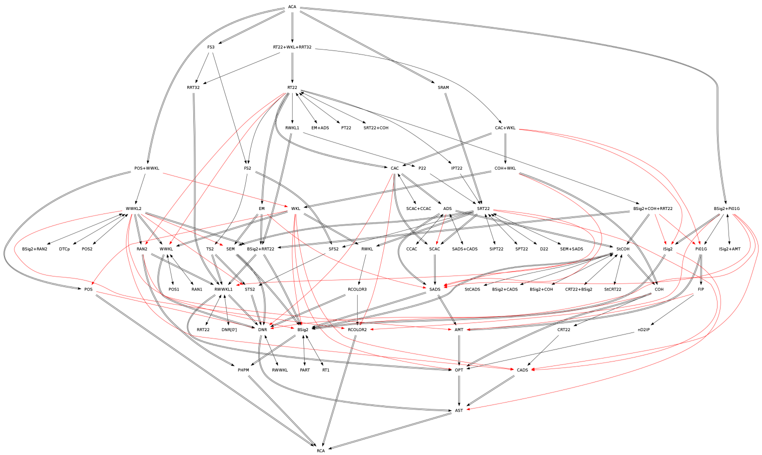
Δ^0_1 Comprehension scheme

$$\forall n(\varphi(n) \Leftrightarrow \psi(n)) \Rightarrow \exists X.\forall n.(x \in X \Leftrightarrow \varphi(n))$$

where $\varphi(n)$ is any Σ^0_1 formula of L_2 in which X does not occur freely and $\psi(n)$ is any Π^0_1 formula of L_2.

The "Big Five" subsystems

Reverse mathematics zoo



ω-structure

Definition (ω-structure)

$$\mathcal{M}_S = (\omega, S, +_\omega, \times_\omega, <_\omega)$$

Example (Minimal ω-model of $\mathsf{RCA_0}$)

$COMP$ is the ω-structure where

$$S = \{X \in 2^\omega : X \text{ is computable}\}$$

Plan

No randomized algorithm property

Definition

Let $\vec{X}_i$ be a sequence of sets. $COMP(\vec{X}_i)$ is the ω-structure where

$$S = \bigcup_{i \in \omega} \{Y : Y \leq_T X_0 \oplus \cdots \oplus X_i\}.$$

Question

Fix a system P *and pick a sequence* $\vec{X}_i$ *at random.*
What is the probability that $COMP(\vec{X}_i) \models \mathsf{P}$ *?*

No randomized algorithm property

Definition

A system P has the *no randomized algorithm property* if when picking a sequence of sets $\vec{X}_i$, the probability that $COMP(\vec{X}_i) \models \mathsf{P}$ is null.

Question

Which systems have the NRA property ?

No randomized algorithm property

Why *no randomized algorithm property ?*

- Consider a principle $\mathsf{P} = \forall Y \exists Z \Phi(Y, Z)$.
- If P has the NRA property, then for almost every sequence $\vec{X}_i$ there is a $Y \in COMP(\vec{X}_i)$ such that no probabilistic algorithm computes a Z such that $\Phi(Y, Z)$.

No randomized algorithm property

n-RAN (n-randomness)

For every X, there is a set Y which is n-random relative to X.

n-WWKL (n-weak weak König's lemma)

Every subtree of $2^{<\omega}$ of positive measure computable in $\emptyset^{(n-1)}$ has an infinite path.

Theorem (Avigdad, Dean & Rute)

For every standard n,

$$\mathsf{RCA_0} + \mathsf{B}\Sigma_n \vdash n\text{-}\mathsf{RAN} \leftrightarrow \mathsf{n\text{-}WWKL}$$

No randomized algorithm property

Theorem

If a system S *has the NRA property*

$$\forall n \quad \mathsf{RCA_0} \nvdash n\text{-}\mathsf{WWKL} \rightarrow \mathsf{S}$$

Proof.

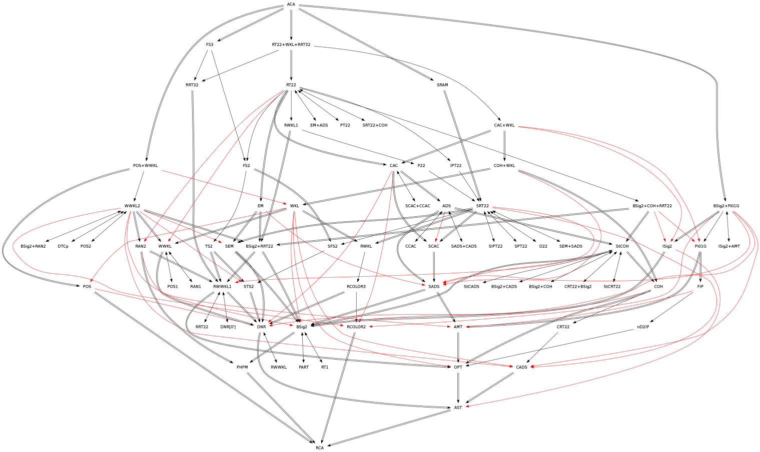
Pick the $\vec{X}_i$ at random. With probability 1, for all i, X_{i+1} is n-random relative to the join of the X_k, $k < i$. Therefore, with probability 1, $COMP(\vec{X}_i)$ is a model of n-WWKL. $\square$

Plan

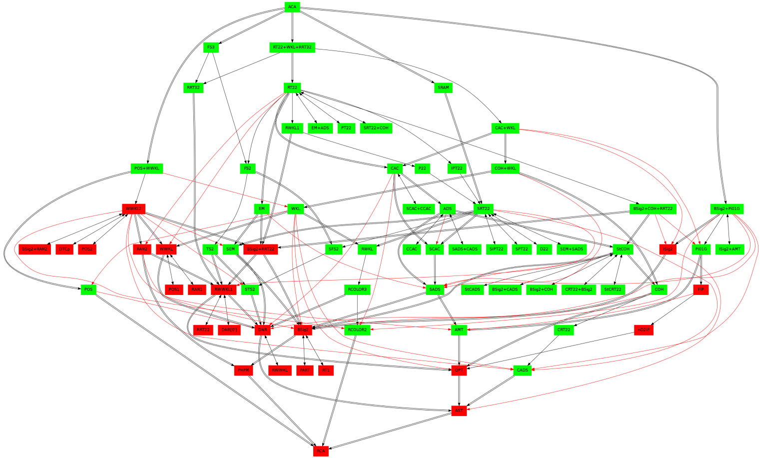
No randomized algorithm property

Which systems have the NRA property ?

We can take the zoo ...



... and classify it



A remark

A lot of (weak) principles have the NRA property

...

Ordering

SADS (Stable ascending descending sequence)

Every linear order of order type $\omega + \omega^*$ has an infinite suborder of order type ω or ω^*.

Theorem (Csima & Mileti)

SADS *has the NRA property*

Proof.

There is a computable linear order of order type $\omega + \omega^*$ such that the measure of oracles computing an infinite suborder of order type ω or ω^* is null. $\square$

Ordering

CADS (Cohesive ascending descending sequence)

Every linear order has a suborder of order type $\omega + \omega^*$ or ω or ω^*.

Theorem (Bienvenu, Patey & Shafer)

CADS *has the NRA property*

Proof.

There is a computable linear order such that the measure of oracles computing an infinite suborder of order type $\omega + \omega^*$ or ω or ω^* is null. $\square$

Genericity

$\Pi^0_1\mathsf{G}$ (Π^0_1 genericity)

Any uniformly Π^0_1 collection of dense sets $D_i \subseteq 2^{<\omega}$ has a G such that $\forall i \exists s (G \upharpoonright s \in D_i)$.

Theorem (Kurtz)

The upward closure of the weakly 2-generic degrees has measure 0.

Theorem (Bienvenu, Patey & Shafer)

$\Pi^0_1\mathsf{G}$ *has the NRA property*

First remark

... but there are non-trivial problems solved by randomness.

Genericity

1-GEN (1-genericity)

For any set X, there exists a set 1-generic relative to X.

Theorem (Kurtz)

Almost every set computes a 1-generic set.

Corollary

1-GEN *does not have the NRA property.*

Rainbow Ramsey Theorem

Definition (k-bounded function)

A coloring function $\mathbb{N}^n \to \mathbb{N}$ is k-bounded if $\mathtt{card}\,\{x \in \mathbb{N}^n : f(x) = c\} \leq k$ for every color c.

RRT^n_k (Rainbow Ramsey Theorem)

For every k-bounded coloring function $f : \mathbb{N}^n \to \mathbb{N}$ there is an infinite set H such that $f \restriction H^n$ is injective.

Rainbow Ramsey Theorem

Theorem (Csima & Mileti)

$\mathsf{RCA}_0 \vdash 2\text{-}\mathsf{RAN} \to \mathsf{RRT}^2_2$

Theorem (Bienvenu, Patey & Shafer)

RRT^3_2 *has the NRA property.*

Proof.

There is a computable 2-bounded coloring $c : [\mathbb{N}]^3 \to \mathbb{N}$ such that the measure of oracles computing an infinite rainbow for c is null. $\square$

Plan

Conclusion

- The following principles have the NRA property: $\Pi^0_1\mathsf{G}$, CADS, SEM, RRT^3_2, POS, $\mathsf{STS}(2)$ RCOLOR_2.
- Any principle below n-WWKL for some n does not have the NRA property.
- This suffices to classify the whole zoo.

Further research

- The NRA property: computing or not a solution with
  - randomness
- What about the ability to compute a solution with
  - randomness
  - other oracles (eg. $0'$)

References

Klaus Ambos-Spies, Bjørn Kjos-Hanssen, Steffen Lempp, and Theodore A Slaman.
Comparing DNR and WWKL.
Journal of Symbolic Logic, pages 1089–1104, 2004.

Jeremy Avigad, Edward T Dean, and Jason Rute.
Algorithmic randomness, reverse mathematics, and the dominated convergence theorem.
Annals of Pure and Applied Logic, 2012.

CT Chong, Theodore A Slaman, and Yue Yang.
The metamathematics of stable Ramsey's theorem for pairs.
To appear.

Stephen Flood.
Reverse mathematics and a ramsey-type könig's lemma.
Journal of Symbolic Logic, 77(4):1272–1280, 2012.

Jiayi Liu.
RT22 does not imply WKL0.
Journal of Symbolic Logic, 77(2):609–620, 2012.

Henry Towsner Manuel Lerman, Reed Solomon.
Separating principles below Ramsey's Theorem for Pairs.
2013.

Stephen G Simpson.
Subsystems of second order arithmetic, volume 42.
Springer Berlin, 1999.

Questions

Thank you for listening !