Classifications of Ramsey-like theorems

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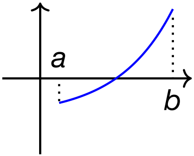
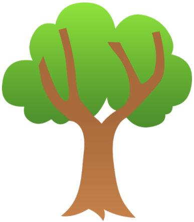
Consider mathematical problems

Intermediate value theorem
For every continuous function f over an interval $[a, b]$ such that $f(a) \cdot f(b) < 0$, there is a real $x \in [a, b]$ such that $f(x) = 0$.



König's lemma
Every infinite, finitely branching tree admits an infinite path.



What functions can problems dominate?

Fix a problem P.

A function $f : \omega \to \omega$ is P-dominated if there is an instance of P such that every solution computes a function dominating f

A function f is hyperimmune if it is not dominated by any computable function.

Ramsey's theorem

$[X]^n$ is the set of unordered n-tuples of elements of X

A k-coloring of $[X]^n$ is a map $f : [X]^n \to k$

A set $H \subseteq X$ is homogeneous for f if $|f([H]^n)| = 1$.

RT^n_k Every k-coloring of $[\mathbb{N}]^n$ admits an infinite homogeneous set.

Pigeonhole principle

RT^1_k Every k-partition of $\mathbb{N}$ admits an infinite part.

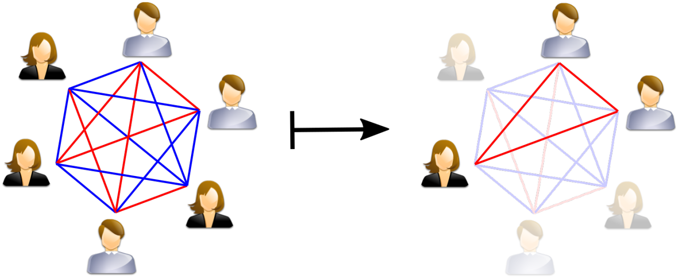
0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	....

$\longmapsto$

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	....

Ramsey's theorem for pairs

RT^2_k Every k-coloring of the infinite clique admits an infinite monochromatic subclique.



Thm (Jockusch)

Every function is RT^2_2-dominated.

Given $g : \omega \to \omega$, an interval $[x, y]$ is *g*-large if $y \geq g(x)$.
Otherwise it is *g*-small.

$$f(x, y) = \begin{cases} 1 & \text{if } [x, y] \text{ is g-large} \\ 0 & \text{otherwise} \end{cases}$$

Thm (Dzhafarov and Jockusch)

No hyperimmune function is RT^1_2-dominated.

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	....

Sparsity of red implies non-sparsity of blue and conversely.

Ramsey's theorem

Over n-tuples

$$\mathrm{RT}^{n}_{k}$$

Using k colors

Ramsey's theorem

Over *n*-tuples

$$\mathrm{RT}^{n}_{k,r}$$

Allows *r* colors

Using *k* colors

Thm (Wang)

No hyperimmune function is $\mathsf{RT}^n_{k,\ell}$-dominated for large ℓ

(whenever ℓ is at least the nth Schröder Number)

Thm (Dorais, Dzhafarov, Hirst, Mileti, Shafer)

Every function is $\mathsf{RT}^n_{k,\ell}$-dominated for small ℓ

(whenever $\ell < 2^{n-1}$)

Erdős-Moser theorem

Fix $f : [\omega]^2 \to 2$.

A set H is transitive if for every $a < b < c \in H$, such that $f(a,b) = f(b,c)$ then $f(a,b) = f(a,c)$.

EM Every 2-coloring of $[\mathbb{N}]^2$ admits an infinite transitive set.

Thm (Jockusch)

Every function is RT^2_2-dominated.

Thm (P.)

No hyperimmune function is EM-dominated.

Is there a maximal weakening P of RT^n_k such that no hyperimmune function is P-dominated?

Ramsey-like theorems

Ramsey-like problems

Fix a formal coloring $\mathtt{f} : [\omega]^n \to k$ and variables $\mathtt{x}_0 < \mathtt{x}_1 < \dots$

An RT^n_k-pattern P is a finite conjunction of formulas

$$\mathtt{f}(\mathtt{x}_{i_1}, \dots \mathtt{x}_{i_n}) = v_1 \wedge \dots \wedge \mathtt{f}(\mathtt{x}_{j_1}, \dots \mathtt{x}_{j_n}) = v_s$$

with $v_1, \dots, v_s < k$

Given a coloring $f : [\omega]^n \to k$, a set $H \subseteq \omega$ f-avoids an RT^n_k-pattern P if $(F, f) \not\models P$ for every finite set $F \subseteq H$.

Example 1

- $\mathtt{f}(\mathtt{x}_0, \mathtt{x}_1) = 0 \land \mathtt{f}(\mathtt{x}_2, \mathtt{x}_3) = 1$
- $\mathtt{f}(\mathtt{x}_0, \mathtt{x}_1) = 1 \land \mathtt{f}(\mathtt{x}_2, \mathtt{x}_3) = 0$
- $\mathtt{f}(\mathtt{x}_0, \mathtt{x}_2) = 0 \land \mathtt{f}(\mathtt{x}_1, \mathtt{x}_3) = 1$
- $\mathtt{f}(\mathtt{x}_0, \mathtt{x}_2) = 1 \land \mathtt{f}(\mathtt{x}_1, \mathtt{x}_3) = 0$
- $\mathtt{f}(\mathtt{x}_0, \mathtt{x}_3) = 0 \land \mathtt{f}(\mathtt{x}_1, \mathtt{x}_2) = 1$
- $\mathtt{f}(\mathtt{x}_0, \mathtt{x}_3) = 1 \land \mathtt{f}(\mathtt{x}_1, \mathtt{x}_2) = 0$

Given a coloring $f : [\omega]^2 \to 2$, a set $H \subseteq \omega$ f-avoids simultaneously all these patterns iff H is f-homogeneous.

Example 2

- $\mathtt{f}(\mathtt{x}_0, \mathtt{x}_1) = 0 \wedge \mathtt{f}(\mathtt{x}_1, \mathtt{x}_2) = 0 \wedge \mathtt{f}(\mathtt{x}_0, \mathtt{x}_2) = 1$
- $\mathtt{f}(\mathtt{x}_0, \mathtt{x}_1) = 1 \wedge \mathtt{f}(\mathtt{x}_1, \mathtt{x}_2) = 1 \wedge \mathtt{f}(\mathtt{x}_0, \mathtt{x}_2) = 0$

Given a coloring $f : [\omega]^2 \to 2$, a set $H \subseteq \omega$ f-avoids simultaneously all these patterns iff H is f-transitive.

Ramsey-like problems

Defi

Given a set V of RT^n_k-patterns, $\mathsf{RT}^n_k(V)$ is the problem whose instances are colorings $f : [\omega]^n \to k$ and solutions are sets f-avoiding every pattern in V.

In particular, RT^n_k, $\mathsf{RT}^n_{k,\ell}$ and EM are Ramsey-like problems.

True Ramsey-like problems

Given problems P and Q, let $\mathsf{P} \leq_{id} \mathsf{Q}$ if $\operatorname{dom} \mathsf{P} \subseteq \operatorname{dom} \mathsf{Q}$, and for every $X \in \operatorname{dom}(\mathsf{P})$, $\mathsf{Q}(X) \subseteq \mathsf{P}(X)$.

A Ramsey-like problem $\mathsf{RT}^n_k(W)$ is true if every instance has a solution.

Lem

A Ramsey-like problem $\mathsf{RT}^n_k(W)$ is true iff $\mathsf{RT}^n_k(W) \leq_{id} \mathsf{RT}^n_k$.

Dominating functions

Thm (P.)

There is a Ramsey-like problem $\mathsf{RT}^n_k(U)$ such that for every Ramsey-like problem $\mathsf{RT}^n_k(W)$, no hyperimmune function is $\mathsf{RT}^n_k(W)$-dominated iff $\mathsf{RT}^n_k(W) \leq_{id} \mathsf{RT}^n_k(U)$.

To decide whether no hyperimmune function is $\mathsf{RT}^n_k(W)$-dominated, simply check that

$$\bigvee W \to \bigvee U$$

is a tautology.

Example: LARGE^2_k

Defi (LARGE^2_k)

For every coloring $f : [\omega]^2 \to k$, there are two colors $s, \ell < k$ and an infinite set $H \subseteq \omega$ such that

- $f[H]^2 \subseteq \{s, \ell\}$
- $f(x,y) = f(y,z) = s$ iff $f(x,z) = s$ for every $x < y < z \in H$

It looks like over H, there is some function $g : \omega \to \omega$ such that

$$f(x,y) = \begin{cases} \ell & \text{if } [x,y] \text{ is g-large} \\ s & \text{otherwise} \end{cases}$$

This analysis generalizes the following theorems:

- RT^2_2 admits avoidance of 1 cone (Seetapun)
- RT^1_2 admits strong avoidance of 1 cone (Dzhafarov and Jockusch)
- EM admits strong avoidance of 1 cone (P.)
- RT^n_{k,C_n} admits strong avoidance of 1 cone (Cholak and P.)
- FS^n admits strong avoidance of 1 cone (Wang)
- ADS does not admit strong avoidance of 1 cone

Further directions

Main goal

Finding an algorithm which, given two sets for RT^n_k patterns U, V, decides whether

- $\mathsf{RT}^n_k(U) \leq_{sc} \mathsf{RT}^n_k(V)$
- $\mathsf{RT}^n_k(U) \leq_{c} \mathsf{RT}^n_k(V)$
- $\mathsf{RT}^n_k(U) \leq_{\omega} \mathsf{RT}^n_k(V)$

Questions

Is there a $\leq_{id}$-maximal Ramsey-like problem P such that

$$\mathsf{RT}^2_2 \not\leq_c \mathsf{P}$$

Is EM this maximal Ramsey-like problem?

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