SRT^2_2 does not imply COH under ω-models

Ludovic Patey
Joint work with Benoit Monin

REVERSE MATHEMATICS

Foundational program that seeks to determine the optimal axioms of ordinary mathematics.

$$\mathsf{RCA}_0 \vdash A \leftrightarrow T$$

in a very weak theory RCA_0
capturing computable mathematics

REVERSE MATHEMATICS

Mathematics are computationally very structured

Almost every theorem is empirically equivalent to one among five big subsystems.

Π^1_1CA

$\downarrow$

ATR

$\downarrow$

ACA

$\downarrow$

WKL

$\downarrow$

RCA_0

REVERSE MATHEMATICS

Mathematics are computationally
very structured

Almost every theorem is empirically equivalent to one among five big subsystems.

Except for Ramsey's theory...

$$
\begin{tikzcd}
 & \Pi^1_1\mathrm{CA} \arrow[d] \\
 & \mathrm{ATR} \arrow[d] \\
 & \mathrm{ACA} \arrow[d] \arrow[dl] \\
\mathrm{RT}^2_2 \arrow[dr] & \mathrm{WKL} \arrow[d] \\
 & \mathrm{RCA}_0
\end{tikzcd}
$$

RAMSEY'S THEOREM

$[X]^n$ is the set of unordered n-tuples of elements of X

A k-coloring of $[X]^n$ is a map $f : [X]^n \to k$

A set $H \subseteq X$ is homogeneous for f if $|f([H]^n)| = 1$.

RT^n_k Every k-coloring of $[\mathbb{N}]^n$ admits an infinite homogeneous set.

Pigeonhole principle

RT^1_k Every k-partition of $\mathbb{N}$ admits an infinite part.

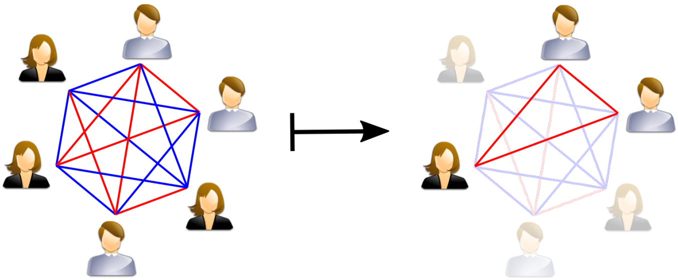
0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	....

$\longmapsto$

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	....

RAMSEY'S THEOREM FOR PAIRS

RT^2_k Every k-coloring of the infinite clique admits an infinite monochromatic subclique.



An infinite set C is $\vec{R}$-cohesive for some sets $R_0, R_1, \dots$ if for every i, either $C \subseteq^* R_i$ or $C \subseteq^* \overline{R}_i$.

COH : Every collection of sets has a cohesive set.

A coloring $f : [\omega]^2 \to 2$ is stable if $\lim_y f(x, y)$ exists for every x.

SRT^2_2 : Every stable coloring of pairs admits an infinite homogeneous set.

$$\mathrm{RCA}_0 \vdash \mathrm{RT}^2_2 \leftrightarrow \mathrm{COH} \wedge \mathrm{SRT}^2_2$$

(Cholak, Jockusch and Slaman)

- Given $f : [\mathbb{N}]^2 \to 2$, define $\langle R_x : x \in \mathbb{N} \rangle$ by $R_x = \{y : f(x, y) = 1\}$
- By COH, there is an $\vec{R}$-cohesive set $C = \{x_0 < x_1 < \dots\}$
- $f : [C]^2 \to 2$ is stable

$$\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \leftrightarrow \mathsf{COH} \wedge \mathsf{SRT}^2_2$$

(Cholak, Jockusch and Slaman)

Thm (Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp, and Slaman)

$\mathsf{RCA}_0 \nvdash \mathsf{COH} \rightarrow \mathsf{SRT}^2_2$

Thm (Chong, Slaman and Yang)

$\mathsf{RCA}_0 \nvdash \mathsf{SRT}^2_2 \rightarrow \mathsf{COH}$

Using a non-standard model containing only low sets.

Thm (Monin and Patey)

$\mathsf{RCA}_0 \nvdash \mathsf{SRT}^2_2 \rightarrow \mathsf{COH}$ on ω-models

Thm (Monin and Patey)

$\mathsf{RCA}_0 \nvdash \mathsf{SRT}^2_2 \to \mathsf{COH}$ on ω-models

Jump cone avoidance

RT^2 admits low_2 solutions

CJS trick: largeness

→

Jump PA avoidance

SRT^2 does not imply COH

Monin's trick: largeness
on product spaces

↑ **Second jump control** ↑

First jump control

Cone avoidance

Seetapun's theorem

CJS trick: overapproximations

→

PA avoidance

Liu's theorem

Liu's trick: valuations

Reformulate the problem using

computability-theoretic

versions of the statements

COH admits a universal instance:

the primitive recursive sets

A set is p-cohesive if it is cohesive for the p.r. sets

Thm (Jockusch and Stephan)

A set is p-cohesive iff its jump is PA over $\emptyset'$

Thm (Jockusch and Stephan)

For every computable sequence of sets $\vec{R}$ and every p-cohesive set C, C computes an $\vec{R}$-cohesive set.

SRT^2_2 can be seen as a Δ^0_2 instance of
the pigeonhole principle

- Given a stable computable coloring $f : [\omega]^2 \to 2$
- Let $A = \{x : \lim_y f(x, y) = 1\}$
- Every infinite set $H \subseteq A$ or $H \subseteq \overline{A}$ computes an infinite f-homogeneous set.

Thm (Monin and Patey)

For every Δ^0_2 set A, there is an infinite set $H \subseteq A$ or $H \subseteq \overline{A}$ whose jump is not of PA degree over $\emptyset'$.

The proof relativizes as follows

Thm (Monin and Patey)

Fix a set Z whose jump is not of PA degree over $\emptyset'$. For every $\Delta^0_2(Z)$ set A, there is an infinite set $H \subseteq A$ or $H \subseteq \overline{A}$ such that $(H \oplus Z)'$ is not of PA degree over $\emptyset'$.

LAYER 1: CONE AVOIDANCE

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STRONG CONE AVOIDANCE

A problem P admits strong cone avoidance if for every non-computable set C and every P-instance X, there is a solution Y to X such that $C \not\leq_T Y$.

Thm (Dzhafarov, Jockusch)

RT^1_2 admits strong cone avoidance

Forcing question

$$c \mathrel{?}\!\vdash \varphi(G)$$

where $c \in \mathbb{P}$ and $\varphi(G)$ is Σ^0_1

Lem

Let $c \in \mathbb{P}$ and $\varphi(G)$ be a Σ^0_1 formula.

(a) If $c \mathrel{?}\!\vdash \varphi(G)$, then $d \Vdash \varphi(G)$ for some $d \leq c$;

(b) If $c \mathrel{?}\!\nvdash \varphi(G)$, then $d \Vdash \neg\varphi(G)$ for some $d \leq c$.

Suppose $c \mathrel{?\vdash} \varphi(G)$ is uniformly Σ^0_1 whenever $\varphi(G)$ is Σ^0_1

Lem

For every non-computable set C and Turing functional Φ_e, the following set is dense in $(\mathbb{P}, \leq)$.

$$D = \{c \in \mathbb{P} : c \Vdash \Phi_e^G \neq C\}$$

Given $c \in \mathbb{P}$, define the Σ^0_1 set $W = \{(x, v) : c \mathrel{?\vdash} \Phi_e^G(x) \downarrow = v\}$

There is some x such that $(x, 1 - C(x)) \in W$ or $(x, C(x)) \notin W$ otherwise we compute C.

PIGEONHOLE FORCING

$$(F_0, F_1, X)$$

Initial segment

Reservoir

- F_i is finite, X is infinite, $\max F_i < \min X$ (Mathias condition)
- $X \in \mathcal{M} \models \mathsf{WKL}$ (Weakness property)
- $F_i \subseteq A_i$ (Combinatorics)

FORCING RELATION

$$(F_0, F_1, X) \Vdash_i \varphi(G_i)$$

Condition

Formula

Σ^0_1 $(F_0, F_1, X) \Vdash_i (\exists x)\varphi(G_i, x)$ if $(\exists w \in \omega)\varphi(F_i, w)$

Π^0_1 $(F_0, F_1, X) \Vdash_i (\forall x)\varphi(G_i, x)$ if $(\forall E \subseteq X)(\forall w)\varphi(F_i \cup E, w)$

FORCING QUESTION

$$(F_0, F_1, X) \mathrel{?}\!\vdash \varphi_0(G_0) \vee \varphi_1(G_1)$$

Lem

Let $c \in \mathbb{P}$ and $\varphi_0(G), \varphi_1(G)$ be a Σ^0_1 formulas.

(a) If $c \mathrel{?}\!\vdash \varphi_0(G_0) \vee \varphi_1(G_1)$, then $d \Vdash_i \varphi_i(G_i)$

(b) If $c \mathrel{?}\!\nvdash \varphi_0(G_0) \vee \varphi_1(G_1)$, then $d \Vdash_i \neg\varphi_i(G_i)$

for some $d \leq c$ and $i < 2$.

CJS TRICK: OVER-APPROXIMATIONS

$$(F_0, F_1, X) \mathrel{?\vdash} \varphi_0(G_0) \vee \varphi_1(G_1)$$

is the $\Sigma^{0,X}_1$ formula

$$(\forall B_0 \sqcup B_1 = X)(\exists i < 2)(\exists E \subseteq B_i)\varphi_i(F_i \cup E)$$

CJS TRICK: OVER-APPROXIMATIONS

- Case 1: $(F_0, F_1, X) \mathrel{?{\vdash}} \varphi_0(G_0) \vee \varphi_1(G_1)$

  Letting $B_i = A_i$, there is an extension $d \leq c$ such that

  $$d \Vdash^0 \varphi_1(G_0) \quad \text{or} \quad d \Vdash^1 \varphi_1(G_1)$$

- Case 2: $(F_0, F_1, X) \mathrel{?{\nvdash}} \varphi_0(G_0) \vee \varphi_1(G_1)$

  The class $\mathcal{C}$ of all $B_0 \sqcup B_1 = \mathbb{N}$ such that

  $$(\forall i < 2)(\forall E_i \subseteq X \cap B_i)\Phi_{e_i}^{F_i \cup E_i}(x) \neq n$$

  is a non-empty $\Pi_1^{0,X}$ class. Pick $B_0 \sqcup B_1 \in \mathcal{C} \cap \mathcal{M}$.

  $$(F_0, F_1, X \cap B_i) \Vdash_i \neg\varphi_i(G_i)$$

LAYER 2: PA AVOIDANCE

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STRONG PA AVOIDANCE

A problem P admits strong PA avoidance if for every P-instance X, there is a solution Y to X of non-PA degree.

Thm (Liu)

RT^1_2 admits strong PA avoidance

Forcing question

$$c \mathrel{?}\!\vdash \varphi(G)$$

where $c \in \mathbb{P}$ and $\varphi(G)$ is Σ^0_1

Lem

Let $c \in \mathbb{P}$ and $\varphi(G)$ and $\psi(G)$ be a Σ^0_1 formulas. If $c \mathrel{?}\!\nvdash \varphi(G)$ and $c \mathrel{?}\!\nvdash \psi(G)$, then $d \Vdash \neg\varphi(G) \wedge \neg\psi(G)$ for some $d \leq c$.

Suppose $c \mathrel{?\!\vdash} \varphi(G)$ is uniformly Σ^0_1 whenever $\varphi(G)$ is Σ^0_1

Lem

For every Turing functional Φ_e, the following set is dense in $(\mathbb{P}, \leq)$.

$$D = \{c \in \mathbb{P} : c \Vdash \Phi^G_e \text{ is not a } DNC_2 \text{ function } \}$$

Given $c \in \mathbb{P}$, define the Σ^0_1 set $W = \{(x, v) : c \mathrel{?\!\vdash} \Phi^G_e(x) \downarrow = v\}$

There is some x such that $(x, \Phi_x(x) \downarrow) \in W$ or $(x, 0), (x, 1) \not\in W$ otherwise we compute a DNC_2 function.

PIGEONHOLE FORCING

$$(F_0, F_1, X)$$

Initial segment

Reservoir

- F_i is finite, X is infinite, $\max F_i < \min X$ (Mathias condition)
- $X \in \mathcal{M} \models \text{RWKL}$ (Weakness property)
- $F_i \subseteq A_i$ (Combinatorics)

FORCING QUESTION

$$(F_0, F_1, X) \mathrel{?}\!\vdash \varphi_0(G_0) \vee \varphi_1(G_1)$$

Lem

Let $c \in \mathbb{P}$ and $\varphi_0(G), \varphi_1(G), \psi_0(G), \psi_1(G)$ be a Σ^0_1 formulas. If $c \mathrel{?}\!\nvdash \varphi_0(G_0) \vee \varphi_1(G_1)$ and $c \mathrel{?}\!\nvdash \psi_0(G_0) \vee \psi_1(G_1)$, then

$$d \Vdash_i \neg\varphi_i(G_i) \text{ and } d \Vdash^j \neg\psi_j(G_j)$$

for some $d \leq c$ and $i, j < 2$.

Bad case: $i \neq j$

Liu's trick: valuations

A valuation is a finite function $p \subseteq \omega \to 2$

A valuation is correct if $p(x) = \Phi_x(x)\downarrow$ for every $x \in \mathrm{dom}(p)$

Two valuations p and q are incompatible if $p(x) \neq q(x)$ for some $x \in \mathrm{dom}(p) \cap \mathrm{dom}(q)$

Lem (Liu)

Let W be a c.e. set of valuations. Either W contains a correct valuation, or for every $k \in \omega$, there are k pairwise incompatible valuations outside W.

LIU'S TRICK: VALUATIONS

Consider the $\Sigma_1^{0,X}$ set of valuations

$$W = \{p : c \mathrel{?\vdash} p \text{ incompatible with either } \Phi_{e_0}^{G_0} \text{ or } \Phi_{e_1}^{G_1}\}$$

Assuming X is not of PA degree, either there is a correct valuation $p \in W$, or three pairwise incorrect valuations $p_0, p_1, p_2 \notin W$.

In the latter case, there is an extension $d \leq c$ and some $i_0, i_1, i_2 < 2$ such that $d \Vdash^{i_s} p_s$ compatible with $\Phi_{e_{i_s}}^{G_{i_s}}$ for each $s < 3$.

LAYER 3: JUMP CONE AVOIDANCE

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STRONG JUMP CONE AVOIDANCE

A problem P admits strong jump cone avoidance if for every non-Δ^0_2 set C and every P-instance X, there is a solution Y to X such that C is not $\Delta^0_2(Y)$.

Thm (Monin, Patey)

RT^1_2 admits strong jump cone avoidance

Forcing question

$$c \mathrel{?}\!\vdash \varphi(G)$$

where $c \in \mathbb{P}$ and $\varphi(G)$ is Σ^0_2

Lem

Let $c \in \mathbb{P}$ and $\varphi(G)$ be a Σ^0_2 formula.

(a) If $c \mathrel{?}\!\vdash \varphi(G)$, then $d \Vdash \varphi(G)$ for some $d \leq c$;

(b) If $c \mathrel{?}\!\nvdash \varphi(G)$, then $d \Vdash \neg\varphi(G)$ for some $d \leq c$.

Suppose $c \mathrel{?}\!\vdash \varphi(G)$ is uniformly Σ^0_2 whenever $\varphi(G)$ is Σ^0_2

Lem

For every non-Δ^0_2 set C and Turing functional Φ_e, the following set is dense in $(\mathbb{P}, \leq)$.

$$D = \{c \in \mathbb{P} : c \Vdash \Phi_e^{G'} \neq C\}$$

Given $c \in \mathbb{P}$, define the Σ^0_2 set $W = \{(x, v) : c \mathrel{?}\!\vdash \Phi_e^{G'}(x) \downarrow= v\}$

There is some x such that $(x, 1 - C(x)) \in W$ or $(x, C(x)) \notin W$ otherwise we $\emptyset'$-compute C.

PIGEONHOLE FORCING

$$(F_0, F_1, X)$$

Initial segment

Reservoir

- F_i is finite, X is infinite, $\max F_i < \min X$ (Mathias condition)
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- $F_i \subseteq A_i$ (Combinatorics)

FORCING Π^0_2 STATEMENTS

Fix a Δ_0 formula $\psi(G, x, y)$ and a condition c.

$$c \Vdash (\forall x)(\exists y)\psi(G, x, y)$$

$$\equiv$$

$$(\forall x)(\forall d \leq c)(\exists e \leq d) e \Vdash (\exists y)\psi(G, x, y)$$

The quantifier $\forall(E_0, E_1, Y) \leq (F_0, F_1, X)$ is $\Pi^0_3(\mathcal{M})$

CJS TRICK: LARGENESS

- Case 1: $(F_0, F_1, X) \mathrel{?}\!\vdash \varphi_0(G_0) \vee \varphi_1(G_1)$

  Letting $B_i = A_i$, there is an extension $d \leq c$ such that

  $$d \Vdash^0 \varphi_1(G_0) \quad \text{or} \quad d \Vdash^1 \varphi_1(G_1)$$

- Case 2: $(F_0, F_1, X) \mathrel{?}\!\nvdash \varphi_0(G_0) \vee \varphi_1(G_1)$

  The class $\mathcal{C}$ of all $B_0 \sqcup B_1 = \mathbb{N}$ such that

  $$(\forall i < 2)(\forall E_i \subseteq X \cap B_i)\Phi_{e_i}^{F_i \cup E_i}(x) \neq n$$

  is a non-empty $\Pi_1^{0,X}$ class. Pick $B_0 \sqcup B_1 \in \mathcal{C} \cap \mathcal{M}$.

  $$(F_0, F_1, X \cap B_i) \Vdash_i \neg\varphi_i(G_i)$$

CJS TRICK: LARGENESS

The only operations on the reservoirs are partitionning and trimming.

Definition

A class $\mathcal{L} \subseteq 2^\omega$ is large if it satisfies the following properties:

(1) For every $X \in \mathcal{L}$ and $Y \supseteq X$, $Y \in \mathcal{L}$

(2) For every $k \in \omega$ and every $X_0 \cup \dots \cup X_{k-1} = \omega$, there is some $j < k$ such that $X_j \in \mathcal{L}$

Fix a countable coded Scott set $\mathcal{M} = \{X_0, X_1, \dots\}$

Fix a sequence of upward-closed Σ^0_1 classes $\mathcal{U}_0, \mathcal{U}_1, \dots \subseteq 2^\omega$

Given $C \subseteq \omega^2$, let

$$\mathcal{U}_C^{\mathcal{M}} = \bigcap_{(i,e)\in C} \mathcal{U}_e^{X_i}$$

Lem

If $\mathcal{U}_C^{\mathcal{M}}$ is not large, then there is some $X_0 \cup \dots \cup X_{k-1} = \omega$ in $\mathcal{M}$ such that for every $i < k$, $X_i \notin \mathcal{U}_C^{\mathcal{M}}$.

Given $X \subseteq \omega$, let $\mathcal{L}_X = \{ Y : Y \cap X \text{ is infinite } \}$.

$$(F_0, F_1, X, C)$$

Initial segment — Reservoir — Largeness

- F_i is finite, $\max F_i < \min X$ (Mathias condition)
- $\mathcal{U}_C^{\mathcal{M}}$ is large, $\mathcal{U}_C^{\mathcal{M}} \subseteq \mathcal{L}_X$ (Second jump control)
- $X \in \mathcal{M} \models \mathrm{WKL}$, C is Δ_2^0 (Weakness property)
- $F_i \subseteq A_i$ (Combinatorics)

Forcing Π^0_2 formulas

Fix a Δ^0_1 formula $\Phi_e(G, x, y)$

$$(F_0, F_1, X, C) \Vdash_i (\forall x)(\exists y)\neg\Phi_e(G_i, x, y)$$

$$\equiv$$

$$(\forall x)(\forall E \subseteq X \cap A^i)(0, \zeta(e, F_i \cup E, x)) \in C$$

where $\mathcal{U}^{X_0}_{\zeta(e,F,x)} = \{Y : (F, Y) \nVdash (\forall y)\Phi_e(G, x, y)\}$

FORCING Π^0_2 FORMULAS

Suppose

$$(F_0, F_1, X, C) \Vdash_i (\forall x)(\exists y)\neg\Phi_e(G_i, x, y)$$

Then

$$(\forall x)(\forall E \subseteq X \cap A^i)(0, \zeta(e, F_i \cup E, x)) \in C$$

Fix some

$$(E_0, E_1, Y, D) \leq (F_0, F_1, X, C)$$

For every x,

$$\mathcal{U}^{\mathcal{M}}_D \subseteq \mathcal{U}^{\mathcal{M}}_C \subseteq \{Z : (E_i, Z) \not\Vdash (\forall y)\Phi_e(G, x, y)\}$$

So

$$(E_i, Z) \not\Vdash (\forall y)\Phi_e(G, x, y)$$

VALIDITY

A condition (F_0, F_1, X, C) is i-valid if $X \cap A^i \in \mathcal{U}_C^{\mathcal{M}}$.

Every condition is either 0-valid or 1-valid

Lem

If (F_0, F_1, X, C) is i-valid and $(F_i, X) \nVdash (\forall y)\Phi_e(G, y)$ then there is an extension (E_0, E_1, Y, C) with $(E_i, Y) \Vdash (\exists y)\neg\Phi_e(G, y)$

FORCING QUESTION

$$(F_0, F_1, X, C) \mathrel{?\!\vdash_i} (\exists x)(\forall y)\Phi_e(G_i)$$

$$\equiv$$

$$\mathcal{U}_C^{\mathcal{M}} \bigcap_{x\in\omega, E\subseteq X\cap A^i} \{Z : (F_i \cup E, Z) \not\Vdash (\forall y)\Phi_e(G, x, y)\} \text{ is not large}$$

Lem

Let $c \in \mathbb{P}$, $i < 2$ and $\varphi(G)$ be a Σ^0_2 formula.

(a) If $c \mathrel{?\!\vdash_i} \varphi(G_i)$, then $d \Vdash_i \varphi(G_i)$ for some $d \leq c$;

(b) If $c \mathrel{?\!\nvdash_i} \varphi(G_i)$, then $d \Vdash_i \neg\varphi(G_i)$ for some $d \leq c$.

JUMP CONE AVOIDANCE: SUMMARY

Let $\mathcal{F}$ be a sufficiently generic filter for the pigeonhole forcing.

By the non-disjunctive Σ^0_2 forcing question, for every $i < 2$ and every $e \in \omega$, there is a condition $c \in \mathcal{F}$ such that $c \Vdash_i \Phi_e^{G'} \neq C$.

For every Σ^0_2 formula $\varphi(G)$, whenever $c \Vdash_i \varphi(G_i)$, then $\varphi(G_i)$ holds.

There is some $i < 2$ such that every condition in $\mathcal{F}$ is i-valid.

Therefore, there is some $i < 2$ such that for every Π^0_2 formula $\varphi(G)$, whenever $c \Vdash_i \varphi(G_i)$, then $\varphi(G_i)$ actually holds.

LAYER 4: JUMP PA AVOIDANCE

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Thm (Monin and Patey)

SRT^2_2 admits jump PA avoidance

Forcing question

$$c \mathrel{?}\vdash \varphi(G)$$

where $c \in \mathbb{P}$ and $\varphi(G)$ is Σ^0_2

Lem

Let $c \in \mathbb{P}$ and $\varphi(G)$ and $\psi(G)$ be a Σ^0_2 formulas. If $c \mathrel{?}\nvdash \varphi(G)$ and $c \mathrel{?}\nvdash \psi(G)$, then $d \Vdash \neg\varphi(G) \wedge \neg\psi(G)$ for some $d \leq c$.

Suppose $c \mathrel{?\!\vdash} \varphi(G)$ is uniformly Σ^0_2 whenever $\varphi(G)$ is Σ^0_2

Lem

For every Turing functional Φ_e, the following set is dense in $(\mathbb{P}, \leq)$.

$$D = \{c \in \mathbb{P} : c \Vdash \Phi_e^{G'} \text{ is not a } DNC_2^{\emptyset'} \text{ function } \}$$

Given $c \in \mathbb{P}$, define the Σ^0_2 set $W = \{(x, v) : c \mathrel{?\!\vdash} \Phi_e^{G'}(x) \downarrow = v\}$

There is some x such that $(x, \Phi_x(x) \downarrow) \in W$ or $(x, 0), (x, 1) \notin W$ otherwise we $\emptyset'$-compute a $\text{DNC}_2^{\emptyset'}$ function.

$$(F_0, F_1, X, C)$$

Initial segment Reservoir Largeness

- F_i is finite, $\max F_i < \min X$ (Mathias condition)
- $\mathcal{U}_C^{\mathcal{M}}$ is large, $\mathcal{U}_C^{\mathcal{M}} \subseteq \mathcal{L}_X$ (Second jump control)
- $X \in \mathcal{M} \models \mathrm{WKL}$, C is Δ^0_2 (Weakness property)
- $F_i \subseteq A_i$ (Combinatorics)

FORCING QUESTION

$$(F_0, F_1, X, C) \mathrel{?{\vdash}_i} (\exists x)(\forall y)\Phi_e(G_i)$$

$$\equiv$$

$$\mathcal{U}_C^{\mathcal{M}} \bigcap_{x\in\omega, E\subseteq X\cap A^i} \{Z : (F_i \cup E, Z) \not\Vdash (\forall y)\Phi_e(G, x, y)\} \text{ is not large}$$

Suppose $c \mathrel{?{\nvdash}_i} \varphi(G_i)$ and $c \mathrel{?{\nvdash}_i} \psi(G_i)$

Then there are two sets $D_0, D_1 \subseteq \omega^2$ such that

$$(F_0, F_1, X, D_0) \Vdash \neg\varphi(G_i) \text{ and } (F_0, F_1, X, D_1) \Vdash \neg\varphi(G_i)$$

But $\mathcal{U}_{D_0}^{\mathcal{M}} \cap \mathcal{U}_{D_1}^{\mathcal{M}}$ may not be large!

MONIN'S TRICK: LARGENESS IN PRODUCT SPACES

Suppose $c\,?\!\nVdash_i \varphi(G_i)$ and $c\,?\!\nVdash_i \psi(G_i)$

Then there are two sets $D_0, D_1 \subseteq \omega^2$ such that

$$(F_0, F_1, X, D_0) \Vdash \neg\varphi(G_i) \text{ and } (F_0, F_1, X, D_1) \Vdash \neg\varphi(G_i)$$

Define a new kind of condition

$$(F_0, F_1, Y_0, Y_1, E)$$

where $Y_0 \subseteq X$, $Y_1 \subseteq X$ and $\mathcal{U}_E^{\mathcal{M}} \subseteq \mathcal{U}_{D_0}^{\mathcal{M}} \times \mathcal{U}_{D_1}^{\mathcal{M}}$

Think of it as $(F_0, F_1, Y_0 \cup Y_1)$ where Y_0 and Y_1 must be dependently large

LARGENESS IN PRODUCT SPACES

Definition

A class $\mathcal{L} \subseteq 2^\omega \times \cdots \times 2^\omega$ is large if it satisfies the following properties:

(1) For every $\langle X_i : i < k \rangle \in \mathcal{L}$ and $Y_i \supseteq X_i$, $\langle Y_i : i < k \rangle \in \mathcal{L}$

(2) For every $\ell \in \omega$ and every $X_0 \cup \cdots \cup X_{\ell-1} = \omega$, there is some $j_0, \ldots, j_{k-1} < \ell$ such that $\langle X_{j_i} : i < k \rangle \in \mathcal{L}$

$$(F_0, F_1, X_0, \dots, X_{k-1}, C)$$

Initial segment — Reservoirs — Largeness

- F_i is finite, $\max F_i < \min(X_0, \dots, X_{k-1})$ (Mathias condition)
- $\mathcal{U}_C^{\mathcal{M}}$ is large, $\mathcal{U}_C^{\mathcal{M}} \subseteq \mathcal{L}_{X_0} \times \cdots \times \mathcal{L}_{X_{k-1}}$ (Second jump control)
- $X_0, \dots, X_{k-1} \in \mathcal{M} \models \mathrm{WKL}$, C is Δ_2^0 (Weakness property)
- $F_i \subseteq A_i$ (Combinatorics)

PARTIAL ORDER

$$(E_0, E_1, Y_0, \ldots, Y_{nk-1}, D) \leq (F_0, F_1, X_0, \ldots, X_{k-1}, C)$$

- $Y_{ks+j} \subseteq X_j$ for every $s < n$ and $j < k$
- $F_i \subseteq E_i$ and $E_i \setminus F_i \subseteq X_0 \cup \cdots \cup X_{k-1}$
- for every $\langle Z_{ks+j} : s < n, j < k \rangle \in \mathcal{U}_D^{\mathcal{M}}$, and $s < n$, $\langle Z_{ks+j} : j < k \rangle \in \mathcal{U}_C^{\mathcal{M}}$

VALIDITY

A condition $(F_0, F_1, X_0, \dots, X_{k-1}, C)$ is i-valid if,

$$\langle X_j \cap A^i : j < k \rangle \in \mathcal{U}_C^{\mathcal{M}}$$

Some conditions are neither 0-valid nor 1-valid!

By largeness, there are some $i_0, \dots, i_{k-1} < 2$ such that

$$\langle X_j \cap A^{i_j} : j < k \rangle \in \mathcal{U}_C^{\mathcal{M}}$$

LIU'S TRICK: VALUATIONS

A valuation is a finite function $p \subseteq \omega \to 2$

A valuation is $\emptyset'$-correct if $p(x) = \Phi_x^{\emptyset'}(x)\downarrow$ for every $x \in \operatorname{dom}(p)$

Two valuations p and q are incompatible if $p(x) \neq q(x)$ for some $x \in \operatorname{dom}(p) \cap \operatorname{dom}(q)$

Lem (Liu)

Let W be a Σ^0_2 set of valuations. Either W contains a $\emptyset'$-correct valuation, or for every $k \in \omega$, there are k pairwise incompatible valuations outside W.

LIU'S TRICK: VALUATIONS

Fix a condition $c = (F_0, F_1, X, C)$

Consider the $\Sigma^{0,X}_1$ set of valuations

$$W = \{p : c \,?\!\vdash_i p \text{ incompatible with } \Phi_e^{G_i'}\}$$

Assuming X is low, either there is a $\emptyset'$-correct valuation $p \in W$, or three pairwise incorrect valuations $p_0, p_1, p_2 \not\in W$.

In the latter case, there are three sets $D_0, D_1, D_2 \subseteq \omega^2$ such that $\mathcal{U}^{\mathcal{M}}_{D_0}, \mathcal{U}^{\mathcal{M}}_{D_1}$ and $\mathcal{U}^{\mathcal{M}}_{D_2}$ are large, and for each $s < 3$.

$$(F_0, F_1, X, D_s) \Vdash_i p_s \text{ compatible with } \Phi_e^{G_i'}$$

PARALLEL CONDITIONS

Let

$$\mathcal{U}_D^{\mathcal{M}} = \mathcal{U}_{D_0}^{\mathcal{M}} \times \mathcal{U}_{D_1}^{\mathcal{M}} \times \mathcal{U}_{D_2}^{\mathcal{M}}$$

and define

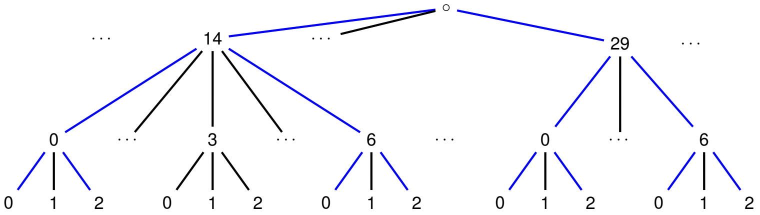
$$d = (F_0^{\{r,s\}}, F_1^{\{r,s\}}, X_0, X_1, X_2, D : r < s < 3)$$

The condition d represents three parallel conditions

- $d^{\{0,1\}} = (F_0^{\{0,1\}}, F_1^{\{0,1\}}, X_0, X_1, \pi_{0,1}(D))$
- $d^{\{0,2\}} = (F_0^{\{0,2\}}, F_1^{\{0,2\}}, X_0, X_2, \pi_{0,2}(D))$
- $d^{\{1,2\}} = (F_0^{\{1,2\}}, F_1^{\{1,2\}}, X_1, X_2, \pi_{1,2}(D))$

One among $d^{\{0,1\}}$, $d^{\{0,2\}}$ and $d^{\{1,2\}}$ is i-valid for some $i < 2$

PARALLEL CONDITIONS



FURTHER RESULTS

Thm (Monin and Patey)

For every Δ^0_2 set A, there is an infinite set $H \subseteq A$ or $H \subseteq \overline{A}$ whose jump is not of 2-random degree.

- Does every set A have an infinite set $H \subseteq A$ or $H \subseteq \overline{A}$ whose jump is not of PA degree relative to $\emptyset'$?
- Does every Δ^0_2 set A have an infinite set $H \subseteq A$ or $H \subseteq \overline{A}$ whose jump is not of DNC degree relative to $\emptyset'$?

CONCLUSION

The computability-theoretic framework about Ramsey's theorem seems to be the good one.

There is hope to answer remaining questions about Ramsey's theorem using the standard combinatorics.

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