Ramsey-like theorems and moduli of computation

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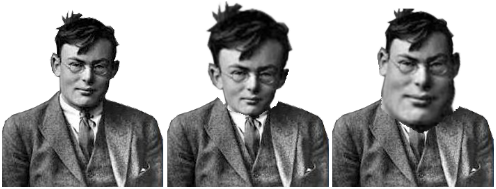
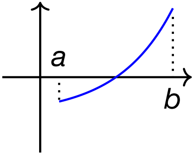
Joint work with Peter Cholak



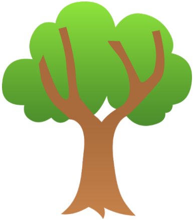
March 11, 2019

Consider mathematical problems

Intermediate value theorem
For every continuous function f over an interval $[a, b]$ such that $f(a) \cdot f(b) < 0$, there is a real $x \in [a, b]$ such that $f(x) = 0$.



König's lemma
Every infinite, finitely branching tree admits an infinite path.



What sets can problems encode?

Fix a problem P.

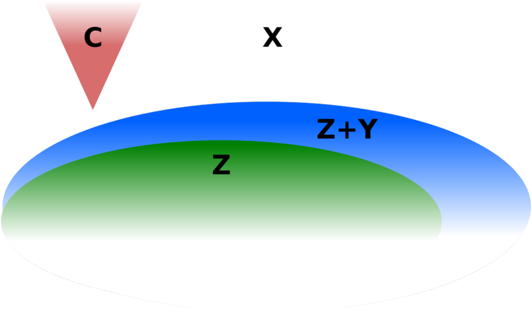
A set S is P-encodable if there is an instance of P such that every solution computes S.

Every computable set is P-encodable.

What sets can problems encode?

Defi (Strong avoidance of 1 cone)

For every Z, every $C \not\leq_T Z$ and every instance X, there is a solution Y such that $C \not\leq_T Z \oplus Y$.



What functions can problems dominate?

Fix a problem P.

A function $f : \omega \to \omega$ is P-dominated if there is an instance of P such that every solution computes a function dominating f

What functions can problems dominate?

A function f is hyperimmune if it is not dominated by any computable function.

Defi (Strong preservation of 1 hyperimmunity)

For every Z, every Z-hyperimmune function f and every instance X, there is a solution Y such that f is $Z \oplus Y$-hyperimmune.

Thm (Downey, Greenberg, Harrison-Trainor, P, Turetsky)

Strong avoidance of 1 cone if and strong preservation of 1 hyperimmunity are equivalent.

Not equivalent in the unrelativized version!

- Fix a non-zero set Y of hyperimmune-free degree.
  Let $\mathsf{P}_1 : Y \mapsto \{Y\}$.
- Fix a hyperimmune f below a Δ^1_1-random.
  Let $\mathsf{P}_2 : f \mapsto \{g : g \geq f\}$.

What sets can encode Ramsey's theorem?

Ramsey's theorem

$[X]^n$ is the set of unordered n-tuples of elements of X

A k-coloring of $[X]^n$ is a map $f : [X]^n \to k$

A set $H \subseteq X$ is homogeneous for f if $|f([H]^n)| = 1$.

RT^n_k Every k-coloring of $[\mathbb{N}]^n$ admits an infinite homogeneous set.

Pigeonhole principle

RT^1_k Every k-partition of $\mathbb{N}$ admits an infinite part.

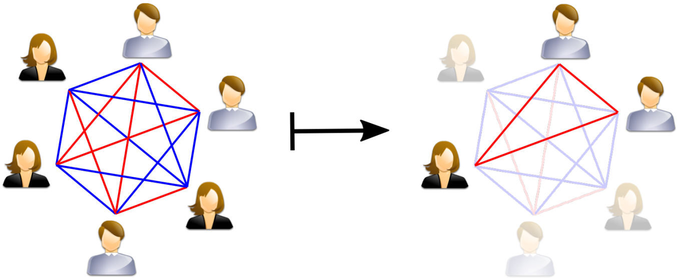
0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	....

$\longmapsto$

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	....

Ramsey's theorem for pairs

RT^2_k Every k-coloring of the infinite clique admits an infinite monochromatic subclique.



Thm (Jockusch)

Every function is RT^2_2-dominated.

Given $g : \omega \to \omega$, an interval $[x, y]$ is g-large if $y \geq g(x)$.
Otherwise it is g-small.

$$f(x, y) = \begin{cases} 1 & \text{if } [x, y] \text{ is g-large} \\ 0 & \text{otherwise} \end{cases}$$

A function f is a modulus of a set S if every function dominating f computes S.

Thm (Groszek and Slaman)

The sets admitting a modulus are the Δ^1_1 sets.

Thm (Jockusch)

Every Δ^1_1 set is RT^2_2-encodable.

A set S is computably encodable if for every infinite set X, there is an infinite subset $Y \subseteq X$ computing S.

Thm (Solovay)

The computably encodable sets are the Δ^1_1 sets.

Thm (Jockusch)

A set is RT^n_k-encodable for some $n \geq 2$ iff it is Δ^1_1.

The encodability power of RT^n_k comes from the

sparsity

of its homogeneous sets.

Thm (Dzhafarov and Jockusch)

The RT^1_2-encodable sets are the computable sets.

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	....

Sparsity of red implies
non-sparsity of blue
and conversely.

Ramsey's theorem

Over n-tuples

$$\mathrm{RT}^{n}_{k}$$

Using k colors

Ramsey's theorem

Over n-tuples

$$\mathsf{RT}^{n}_{k,r}$$

Allows r colors

Using k colors

Thm (Wang)

A set is $\mathsf{RT}^n_{k,\ell}$-encodable iff it is computable for large ℓ

(whenever ℓ is at least the *n*th Schröder Number)

Thm (Dorais, Dzhafarov, Hirst, Mileti, Shafer)

A set is $\mathsf{RT}^n_{k,\ell}$-encodable iff it is Δ^1_1 for small ℓ

(whenever $\ell < 2^{n-1}$)

Thm (Cholak, P.)

Every function is $\mathsf{RT}^n_{k,\ell}$-dominated for $\ell < 2^{n-1}$.

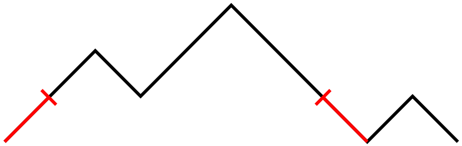
$$f(x_1, x_2, \ldots, x_n) = \langle [x_1, x_2] \text{ g-large?}, \ldots, [x_{n-1}, x_n] \text{ g-large?} \rangle$$

Thm (Cholak, P.)

If a set is $\mathsf{RT}^n_{k,\ell}$-encodable for $\ell \geq 2^{n-1}$ then it is arithmetical.

Catalan numbers

C_n is the number of trails of length $2n$.



$$C_0 = 1 \qquad \text{and} \qquad C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$$

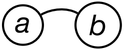
1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786,…

Left-c.e. function

Defi

A function $g : \omega \to \omega$ is left-c.e. if there is a uniformly computable sequence of functions $g_0 \leq g_1 \leq \ldots$ limiting to g.

Given $x_0, \ldots, x_{n-1}$, define the graph of size n by



- if $b = a + 1$ and $[x_a, x_{a+1}]$ is g-large ; or
- if $b > a + 1$ and $[x_a, x_{a+1}]$ is g_{x_b}-small

Defi

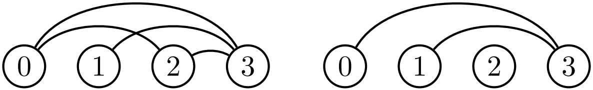
A largeness graph is a pair $(\{0,\dots,n-1\},E)$ such that

(a) If $\{i,i+1\} \in E$, then for every $j > i+1$, $\{i,j\} \notin E$

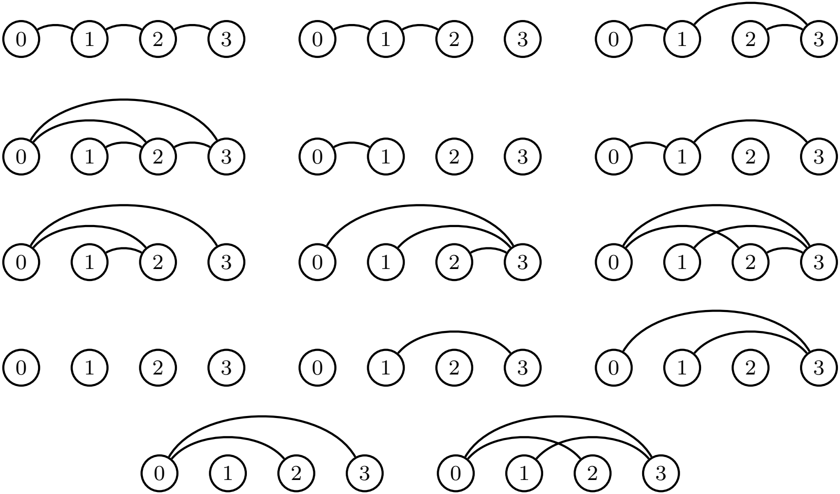
(b) If $i < j < n$, $\{i,i+1\} \notin E$ and $\{j,j+1\} \in E$, then $\{i,j+1\} \in E$

(c) If $i+1 < j < n-1$ and $\{i,j\} \in E$, then $\{i,j+1\} \in E$

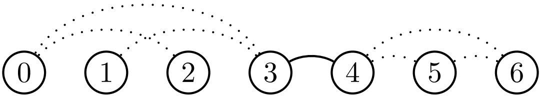
(d) If $i+1 < j < k < n$ and $\{i,j\} \notin E$ but $\{i,k\} \in E$, then $\{j-1,k\} \in E$



Largeness graphs of size 4



Counting largeness graphs



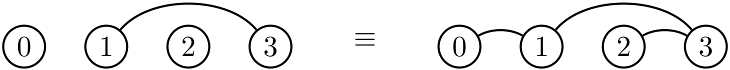
A largeness graph $\mathcal{G} = (\{0, \ldots, n-1\}, E)$ is packed if for every $i < n-2$, $\{i, i+1\} \notin E$.

- L_n = number of largeness graphs of size n
- P_n = number of packed largeness graphs of size n

$$L_0 = 1 \qquad \text{and} \qquad L_{n+1} = \sum_{i=0}^{n} P_{i+1} L_{n-i}$$

Counting packed largeness graphs

A largeness graph $\mathcal{G} = (\{0, \ldots, n-1\}, E)$ of size $n \geq 2$ is normal if $\{n-2, n-1\} \in E$.



Thm (Cholak, P.)

The following are in one-to-one correspondance:

(a) packed largeness graphs of size n

(b) normal largeness graphs of size n

(c) largeness graphs of size $n-1$

Thm (Cholak, P.)

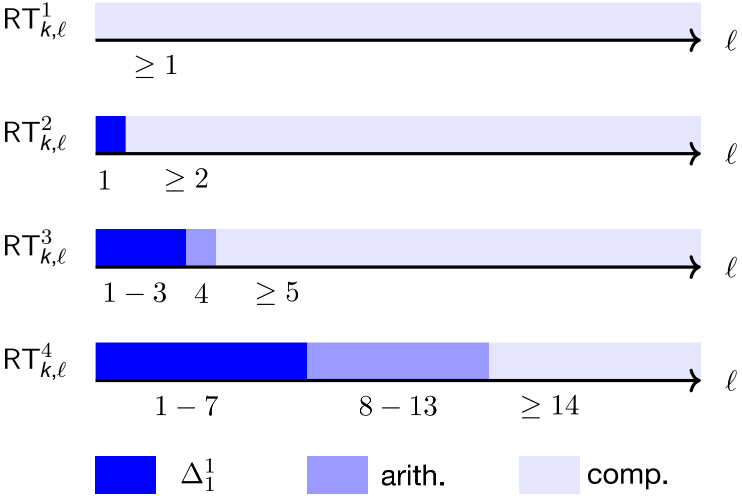
Every left-c.e. function is $\mathrm{RT}^n_{k,\ell}$-dominated for $\ell < C_n$.

$$f(x_1, x_2, \ldots, x_n) = \text{ the largeness graph of g}$$

Thm (Cholak, P.)

The $\mathrm{RT}^n_{k,\ell}$-encodable sets for $\ell \geq C_n$ are the computable sets.

$\mathsf{RT}^n_{k,\ell}$-encodable sets



Ramsey-like theorems

Erdős-Moser theorem

Fix $f : [\omega]^2 \to 2$.

A set H is transitive if for every $a < b < c \in H$, such that $f(a, b) = f(b, c)$ then $f(a, b) = f(a, c)$.

EM Every 2-coloring of $[\mathbb{N}]^2$ admits an infinite transitive set.

Thm (Jockusch)

Every function is RT^2_2-dominated.

Thm (P.)

EM admits strong avoidance of 1 cone.

Is there a maximal weakening of RT^n_k
which admits strong avoidance of 1 cone?

Ramsey-like problems

Fix a formal coloring $\mathtt{f} : [\omega]^n \to k$ and variables $\mathtt{x}_0 < \mathtt{x}_1 < \dots$

An RT^n_k-pattern P is a finite conjunction of formulas

$$\mathtt{f}(\mathtt{x}_{i_1}, \dots \mathtt{x}_{i_n}) = v_1 \wedge \dots \wedge \mathtt{f}(\mathtt{x}_{j_1}, \dots \mathtt{x}_{j_n}) = v_s$$

with $v_1, \dots, v_s < k$

Given a coloring $f : [\omega]^n \to k$, a set $H \subseteq \omega$ f-avoids an RT^n_k-pattern P if $(F, f) \not\models P$ for every finite set $F \subseteq H$.

Ramsey-like problems

Defi

Given a set V of RT^n_k-patterns, $\mathsf{RT}^n_k(V)$ is the problem whose instances are colorings $f : [\omega]^n \to k$ and solutions are sets f-avoiding every pattern in V.

In particular, RT^n_k, $\mathsf{RT}^n_{k,\ell}$ and EM are Ramsey-like problems.

Thm (P.)

For every $n, k \geq 1$, there is a strongest Ramsey-like problem $\mathsf{RT}^n_k(V)$ which admits strong avoidance of 1 cone.

Ramsey-like problems

Given problems P and Q, let $\mathsf{P} \leq_{id} \mathsf{Q}$ if $\operatorname{dom} \mathsf{P} \subseteq \operatorname{dom} \mathsf{Q}$, and for every $X \in \operatorname{dom}(\mathsf{P})$, $\mathsf{Q}(X) \subseteq \mathsf{P}(X)$.

Thm (P.)

There is a Ramsey-like problem $\mathsf{SCA\text{-}RT}^n_k$ such that for every set V of RT^n_k-patterns, $\mathsf{RT}^n_k(V)$ admits strong avoidance of 1 cone iff $\mathsf{RT}^n_k(V) \leq_{id} \mathsf{SCA\text{-}RT}^n_k$.

To decide strong avoidance for $\mathsf{RT}^n_k(V)$, simply check that

$$\bigvee V \to \bigvee V_{\mathsf{SCA\text{-}RT}^n_k}$$

is a tautology.

Example: $\mathsf{SCA\text{-}RT}^2_k$

Defi ($\mathsf{SCA\text{-}RT}^2_k$)

For every coloring $f : [\omega]^2 \to k$, there are two colors $s, \ell < k$ and an infinite set $H \subseteq \omega$ such that

- $f[H]^2 \subseteq \{s, \ell\}$
- $f(x,y) = f(y,z) = s$ iff $f(x,z) = s$ for every $x < y < z \in H$

It looks like over H, there is some function $g : \omega \to \omega$ such that

$$f(x,y) = \begin{cases} \ell & \text{if } [x,y] \text{ is g-large} \\ s & \text{otherwise} \end{cases}$$

An open question

Is there a set X such that
every infinite set $H \subseteq X$ or $H \subseteq \overline{X}$
has a jump of PA degree over $\emptyset'$?

Thm (Monin, P.)

Fix a non-Δ^0_2 set B. For every set X, there is an infinite set $H \subseteq X$ or $H \subseteq \overline{X}$ such that B is not $\Delta^{0,H}_2$.

Conclusion

Ramsey-type problems compute through sparsity.

The computational properties of Ramsey-type problems are consequences of their combinatorics.

A conclusion with two sentences is too short, so here is a third one.

References

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