Ramsey's theorem under a computable perspective

Ludovic PATEY

Motivations

REVERSE MATHEMATICS

Foundational program that seeks to determine the optimal axioms of ordinary mathematics.

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Foundational program that seeks to determine the optimal axioms of ordinary mathematics.

$$\mathsf{RCA}_0 \vdash A \leftrightarrow T$$

in a very weak theory RCA_0
capturing computable mathematics

RCA_0

Robinson arithmetics

$$m+1 \neq 0 \qquad m+0=m$$
$$m+1=n+1 \rightarrow m=n \qquad m+(n+1)=(m+n)+1$$
$$\neg(m<0) \qquad m\times 0=0$$
$$m<n+1 \leftrightarrow (m<n \vee m=n) \qquad m\times(n+1)=(m\times n)+m$$

Σ^0_1 induction scheme

$$\varphi(0) \wedge \forall n(\varphi(n) \Rightarrow \varphi(n+1)) \Rightarrow \forall n \varphi(n)$$

where $\varphi(n)$ is Σ^0_1

Δ^0_1 comprehension scheme

$$\forall n(\varphi(n) \Leftrightarrow \psi(n)) \Rightarrow \exists X \forall n(n \in X \Leftrightarrow \varphi(n))$$

where $\varphi(n)$ is Σ^0_1 with free X, and ψ is Π^0_1.

Σ^0_n induction scheme

$$\varphi(0) \wedge \forall n(\varphi(n) \Rightarrow \varphi(n+1)) \Rightarrow \forall n \varphi(n)$$

where $\varphi(n)$ is Σ^0_n

bounded Δ^0_n comprehension scheme

$$\forall t \forall n(\varphi(n) \Leftrightarrow \psi(n)) \Rightarrow \exists X \forall n(n \in X \Leftrightarrow (x < t \wedge \varphi(n)))$$

where $\varphi(n)$ is Σ^0_n with free X, and ψ is Π^0_n.

REVERSE MATHEMATICS

Mathematics are computationally very structured

Almost every theorem is empirically equivalent to one among five big subsystems.

$$\Pi^1_1\mathrm{CA} \to \mathrm{ATR} \to \mathrm{ACA} \to \mathrm{WKL} \to \mathrm{RCA}_0$$

HILBERT'S PROGRAM

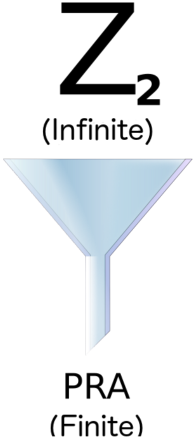
Justification of infinitary methods
to prove finitistic mathematics

Finitistic reductionnism:

$$T \vdash \varphi \Rightarrow PRA \vdash \varphi$$

where φ is a Π^0_1 formula

"At least 85% of mathematics
are reducible to finitistic methods"
(Stephen Simpson



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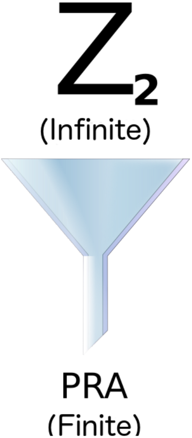

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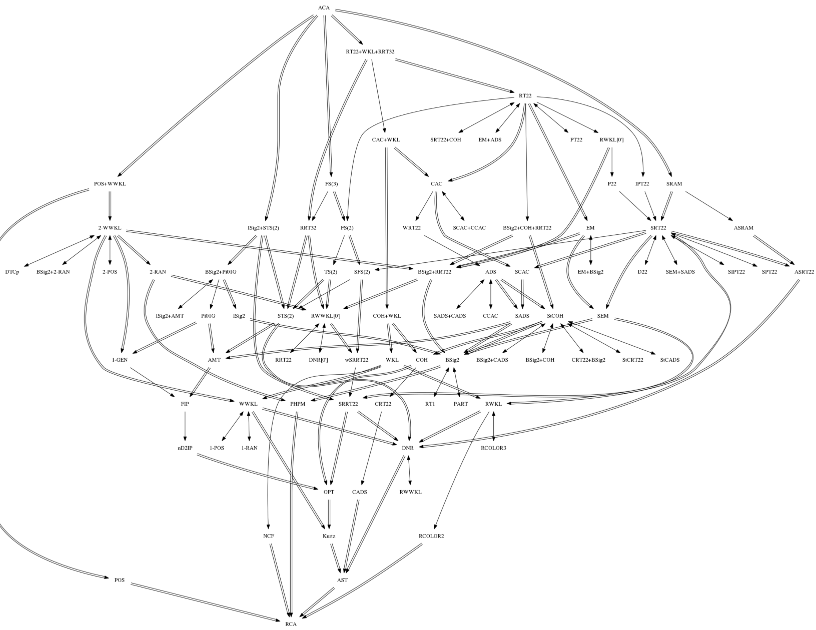
REVERSE MATHEMATICS

Mathematics are computationally very structured

Almost every theorem is empirically equivalent to one among five big subsystems.

Except for Ramsey's theory...

$$\begin{tikzcd}
 & \Pi^1_1\mathrm{CA} \arrow[d] \\
 & \mathrm{ATR} \arrow[d] \\
 & \mathrm{ACA} \arrow[d] \arrow[dl] \\
\mathrm{RT}^2_2 \arrow[dr] & \mathrm{WKL} \arrow[d] \\
 & \mathrm{RCA}_0
\end{tikzcd}$$


ACA
RT22+WKL+RRT32
RT22
CAC+WKL
SRT22+COH
EM+ADS
PT22
RWKL[0']
POS+WWKL
FS(3)
CAC
P22
IPT22
SRAM
2-WWKL
ISig2+STS(2)
RRT32
FS(2)
WRT22
SCAC+CCAC
BSig2+COH+RRT22
EM
SRT22
ASRAM
DTCp
BSig2+2-RAN
2-POS
2-RAN
BSig2+Pi01G
TS(2)
SFS(2)
BSig2+RRT22
ADS
SCAC
EM+BSig2
D22
SEM+SADS
SIPT22
SPT22
ASRT22
ISig2+AMT
Pi01G
ISig2
STS(2)
RWWKL[0']
COH+WKL
SADS+CADS
CCAC
SADS
StCOH
SEM
1-GEN
AMT
RRT22
DNR[0']
wSRRT22
WKL
COH
BSig2
BSig2+CADS
BSig2+COH
CRT22+BSig2
StCRT22
StCADS
FIP
WWKL
PHPM
SRRT22
CRT22
RT1
PART
RWKL
nD2IP
1-POS
1-RAN
DNR
RCOLOR3
OPT
CADS
RWWKL
NCF
Kurtz
RCOLOR2
POS
AST
RCA

What is Ramsey's theorem?

RAMSEY'S THEOREM

$[X]^n$ is the set of unordered n-tuples of elements of X

A k-coloring of $[X]^n$ is a map $f : [X]^n \to k$

A set $H \subseteq X$ is homogeneous for f if $|f([H]^n)| = 1$.

RT^n_k Every k-coloring of $[\mathbb{N}]^n$ admits an infinite homogeneous set.

PIGEONHOLE PRINCIPLE

RT^1_k Every k-partition of $\mathbb{N}$ admits an infinite part.

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
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$\longmapsto$

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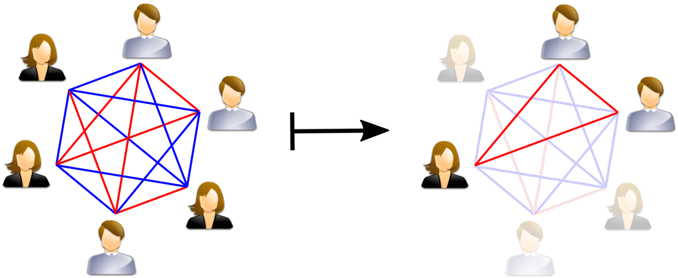
RAMSEY'S THEOREM FOR PAIRS

RT^2_k Every k-coloring of the infinite clique admits an infinite monochromatic subclique.



Reverse mathematics

from a computational viewpoint.

STANDARD MODELS OF RCA_0

An ω-structure is a structure $\mathcal{M} = \{\omega, \mathcal{S}, <, +, \cdot\}$ where

(i) ω is the set of standard natural numbers

(ii) $<$ is the natural order

(iii) $+$ and $\cdot$ are the standard operations over natural numbers

(iv) $\mathcal{S} \subseteq \mathcal{P}(\omega)$

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An ω-structure is fully specified by its second-order part $\mathcal{S}$.

Turing ideal $\mathcal{M}$

- $(\forall X \in \mathcal{M})(\forall Y \leq_T X)[Y \in \mathcal{M}]$
- $(\forall X, Y \in \mathcal{M})[X \oplus Y \in \mathcal{M}]$

Examples

- $\{X : X \text{ is computable }\}$
- $\{X : X \leq_T A \wedge X \leq_T B\}$ for some sets A and B

Let $\mathcal{M} = \{\omega, \mathcal{S}, <, +, \cdot\}$ be an ω-structure

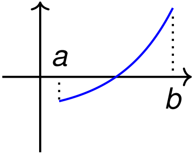
$$\mathcal{M} \models \mathsf{RCA}_0$$

$$\equiv$$

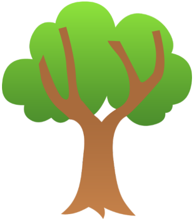
$\mathcal{S}$ is a Turing ideal

Many theorems can be seen as problems.

Intermediate value theorem
For every continuous function f over an interval $[a, b]$ such that $f(a) \cdot f(b) < 0$, there is a real $x \in [a, b]$ such that $f(x) = 0$.



König's lemma
Every infinite, finitely branching tree admits an infinite path.



Let $\mathcal{M}$ be a Turing ideal and P, Q be problems.

Satisfaction

$$\mathcal{M} \models \mathsf{P}$$

if every P-instance in $\mathcal{M}$ has a solution in $\mathcal{M}$.

Computable entailment

$$\mathsf{P} \models_c \mathsf{Q}$$

if every Turing ideal satisfying P satisfies Q.

$$\mathsf{RT}^2_2 \not\models_c \mathsf{ACA}$$

(Seetapun and Slaman, 1995)

- Build $\mathcal{M} \models \mathsf{RT}^2_2$ with $\emptyset' \notin \mathcal{M}$
- If $\mathcal{M} \models \mathsf{ACA}$ then $\emptyset' \in \mathcal{M}$

$$\emptyset' = \{e : (\exists s)\Phi_e(e) \text{ halts after } s \text{ steps } \}$$

Build $\mathcal{M} \models \mathsf{RT}^2_2$ with $\emptyset' \notin \mathcal{M}$.

Thm (Seetapun and Slaman)

Suppose $A \not\leq_T Z$. Then every Z-computable $f : [\omega]^2 \to 2$ has an infinite f-homogeneous set H such that $A \not\leq_T Z \oplus H$.

Start with $\mathcal{M}_0 = \{Z : Z \text{ is computable }\}$. In particular $\emptyset' \notin \mathcal{M}_0$.

Given a Turing ideal $\mathcal{M}_n = \{Z : Z \leq_T U\}$ where $\emptyset' \not\leq_T U$,

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Given a Turing ideal $\mathcal{M}_n = \{Z : Z \leq_T U\}$ where $\emptyset' \not\leq_T U$,

1. pick some $f : [\omega]^2 \to 2$ in $\mathcal{M}_n$

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2. let H be f-homogeneous set such that $\emptyset' \not\leq_T U \oplus H$

$$\text{Build } \mathcal{M} \models \mathsf{RT}^2_2 \text{ with } \emptyset' \notin \mathcal{M}.$$

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1. pick some $f : [\omega]^2 \to 2$ in $\mathcal{M}_n$
2. let H be f-homogeneous set such that $\emptyset' \not\leq_T U \oplus H$
3. let $\mathcal{M}_{n+1} = \{Z : Z \leq_T U \oplus H\}$

Non-implications over RCA_0 often involve purely computability-theoretic arguments.

For $m, n \geq 3$,

$$\mathsf{RCA}_0 \vdash \mathsf{RT}^m_2 \leftrightarrow \mathsf{RT}^n_2$$

(Jockusch)

Theorem (Jockusch)

For every $n \geq 3$, there is a computable coloring $f : [\omega]^n \to 2$ such that every infinite f-homogeneous set computes $\emptyset^{(n-2)}$.

Let $f(x, y, z) = 1$ if the approximation of $\emptyset' \upharpoonright x$ at stage y and at stage z coincide.

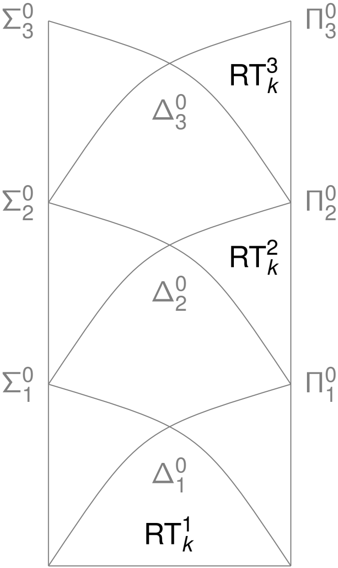
Fix some $n \geq 2$.

Thm (Jockusch)

Every computable instance of RT^n_k has a Π^0_n solution.

Thm (Jockusch)

There is a computable instance of RT^n_k with no Σ^0_n solution.



For $k, \ell \geq 2$,

$$\mathsf{RCA}_0 \vdash \mathsf{RT}^n_k \leftrightarrow \mathsf{RT}^n_\ell$$

Given a coloring $f : [\omega]^n \to \{\mathrm{red}, \mathrm{green}, \mathrm{blue}\}$

- Define $g : [\omega]^n \to \{\mathrm{red}, \mathrm{grue}\}$ by merging green and blue
- Apply RT^n_2 on g to obtain H such that $g[H]^n = \{\mathrm{red}\}$ or $g[H]^n = \{\mathrm{grue}\}$
- In the latter case, apply RT^n_2 on $f[H]^n \to \{\mathrm{green}, \mathrm{blue}\}$ to obtain G such that $f[G]^n = \{\mathrm{green}\}$ or $f[G]^n = \{\mathrm{blue}\}$

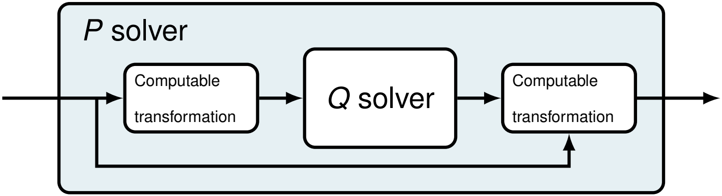
We use more than once the premise for

$$RCA_0 \vdash RT^n_2 \rightarrow RT^{n+1}_2$$

$$RCA_0 \vdash RT^n_k \rightarrow RT^n_{k+1}$$

Can we do it in one step?

COMPUTABLE REDUCTION



$$P \leq_c Q$$

Every P-instance I computes a Q-instance J such that for every solution X to J, $X \oplus I$ computes a solution to I.

$$\mathsf{RT}^{n+1}_2 \not\leq_c \mathsf{RT}^n_2$$

(Jockusch)

- Pick a computable coloring $f : [\omega]^{n+1} \to 2$ with no Σ^0_{n+1} solution
- Every computable coloring $g : [\omega]^n \to 2$ has a Π^0_n solution.

A function $f : \omega \to \omega$ is hyperimmune if it is not dominated by any computable function.

Thm (P.)

There is a computable coloring $f : [\omega]^2 \to k + 1$ and hyperimmune functions $h_0, \dots, h_k$ such that for every infinite f-homogeneous set H, at most one h is H-hyperimmune.

Thm (P.)

Let $h_0, \dots, h_k$ be hyperimmune. For every computable coloring $f : [\omega]^2 \to k$, there is an infinite f-homogeneous set H such that at least two h's are H-hyperimmune.

$$\mathsf{RT}^2_{k+1} \not\leq_c \mathsf{RT}^2_k$$

(P.)

- Pick a computable coloring $f : [\omega]^2 \to k+1$ and hyperimmune functions $h_0, \dots, h_k$ such that for every solution H, at most one h is H-hyperimmune.
- Every computable coloring $g : [\omega]^2 \to k$ has a solution H such that at least two h's are H-hyperimmune.

The naive color-merging proof is optimal with respect to the number of applications in

$$\mathsf{RCA}_0 \vdash \mathsf{RT}^2_k \to \mathsf{RT}^2_\ell$$

PIGEONHOLE PRINCIPLE

RT^1_k Every k-partition of $\mathbb{N}$ admits an infinite part.

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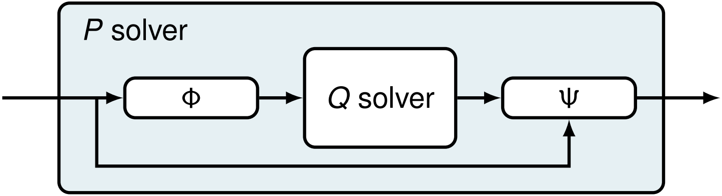
For $k, \ell \geq 2$,

$$\mathsf{RT}^1_k \leq_c \mathsf{RT}^1_\ell$$

No need to use RT^1_ℓ as

RT^1_k is computably true

WEIHRAUCH REDUCTION



$$P \leq_W Q$$

There are Φ and Ψ such that for every P-instance I, Φ^I is a Q-instance such that for every solution X to Φ^I, $\Psi^{X \oplus I}$ is a solution to I.

$$\mathrm{RT}^1_{k+1} \not\leq_W \mathrm{RT}^1_k$$

(Brattka and Rakotoniaina)

Given Φ and Ψ. Build an instance I of RT^1_3. Let

$$I = 000000\ldots$$

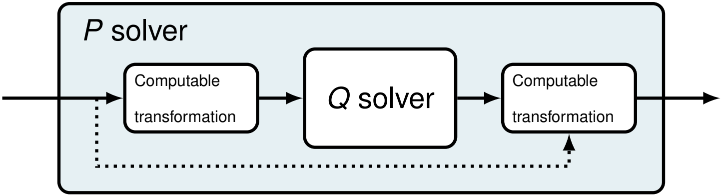
until $\Psi^F(n)\downarrow$ with F of color some $c < 2$ in Φ^I. Then let

$$I = 0000001111111\ldots$$

until $\Psi^G(m)\downarrow$ with G of color $1-c$ in Φ^I. Then let

$$I = 0000001111111222222\ldots$$

STRONG COMPUTABLE REDUCTION



$$\mathrm{P} \leq_{sc} \mathrm{Q}$$

Every P-instance I computes a Q-instance J such that every solution X to J, computes (without I) a solution to I.

A function $f : \omega \to \omega$ is hyperimmune if it is not dominated by any computable function.

Thm (P.)

There is a coloring $f : \omega \to k+1$ and hyperimmune functions $h_0, \dots, h_k$ such that for every infinite f-homogeneous set H, at most one h is H-hyperimmune.

Thm (P.)

Let $h_0, \dots, h_k$ be hyperimmune. For every coloring $f : \omega \to k$, there is an infinite f-homogeneous set H such that at least two h's are H-hyperimmune.

$$\mathrm{RT}^1_{k+1} \not\leq_{sc} \mathrm{RT}^1_k$$

(P.)

- Pick a coloring $f : \omega \to k+1$ and hyperimmune functions $h_0, \dots, h_k$ such that for every solution H, at most one h is H-hyperimmune.
- Every coloring $g : \omega \to k$ has a solution H such that at least two h's are H-hyperimmune.

$$\mathrm{RCA}_0 \vdash \forall k \mathrm{RT}^1_k \leftrightarrow \mathrm{B}\Sigma^0_2$$

(Hirst)

$\mathrm{B}\Sigma^0_2$: For every Σ^0_2 formula φ,

$$(\forall x < t)(\exists y)\varphi(x, y) \rightarrow (\exists u)(\forall x < t)(\exists y < u)\varphi(x, y)$$

”A finite union of finite sets is finite”

What sets can encode Ramsey's theorem?

Fix a problem P.

A set S is P-encodable if there is an instance of P such that every solution computes S.

What sets can encode an instance of RT^n_k?

A function f is a modulus of a set S if every function dominating f computes S.

A set S is computably encodable if for every infinite set X, there is an infinite subset $Y \subseteq X$ computing S.

Thm (Solovay, Groszek and Slaman)

Given a set S, TFAE

- S is computably encodable
- S admits a modulus
- S is hyperarithmetic

Thm (Jockusch)

A set is RT^n_k-encodable for some $n \geq 2$ iff it is hyperarithmetic.

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A set is RT^n_k-encodable for some $n \geq 2$ iff it is hyperarithmetic.

Proof ($\Rightarrow$).

Let $g : [\omega]^n \to k$ be a coloring whose homogeneous sets compute S.

Since every infinite set has a homogeneous subset, S is computably encodable.

Thus S is hyperarithmetic. $\square$

Thm (Jockusch)

A set is RT^n_k-encodable for some $n \geq 2$ iff it is hyperarithmetic.

Proof ($\Leftarrow$).

Let S be hyperarithmetic with modulus μ_S.

Define $g : [\omega]^2 \to 2$ by $g(x, y) = 1$ iff $y > \mu_S(x)$.

Let $H = \{x_0 < x_1 < \dots\}$ be an infinite g-homogeneous set.

The function $p_H(n) = x_n$ dominates μ_S, hence computes S. $\square$

The encodability power of RT^n_k comes from the

sparsity

of its homogeneous sets.

What about RT^1_k?

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	....

Sparsity of red implies non-sparsity of blue and conversely.

Thm (Dzhafarov and Jockusch)

A set is RT^1_2-encodable iff it is computable.

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A set is RT^1_2-encodable iff it is computable.

Input : a set $S \not\leq_T \emptyset$ and a 2-partition $A_0 \sqcup A_1 = \mathbb{N}$

Output : an infinite set $G \subseteq A_i$ such that $S \not\leq_T G$

$$(F_0, F_1, X)$$

Initial segment

Reservoir

- F_i is finite, X is infinite, $\max F_i < \min X$ (Mathias condition)
- $S \not\leq_T X$ (Weakness property)
- $F_i \subseteq A_i$ (Combinatorics)

Extension

$$(E_0, E_1, Y) \leq (F_0, F_1, X)$$

- $F_i \subseteq E_i$
- $Y \subseteq X$
- $E_i \setminus F_i \subseteq X$

Satisfaction

$$\langle G_0, G_1 \rangle \in [F_0, F_1, X]$$

- $F_i \subseteq G_i$
- $G_i \setminus F_i \subseteq X$

$$[E_0, E_1, Y] \subseteq [F_0, F_1, X]$$

$$(F_0, F_1, X) \Vdash \varphi(G_0, G_1)$$

Condition Formula

$\varphi(G_0, G_1)$ holds for every $\langle G_0, G_1 \rangle \in [F_0, F_1, X]$

Input : a set $S \not\leq_T \emptyset$ and a 2-partition $A_0 \sqcup A_1 = \mathbb{N}$

Output : an infinite set $G \subseteq A_i$ such that $S \not\leq_T G$

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$$\Phi_{e_0}^{G_0} \neq S \vee \Phi_{e_1}^{G_1} \neq S$$

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Output : an infinite set $G \subseteq A_i$ such that $S \not\leq_T G$

$$\Phi_{e_0}^{G_0} \neq S \vee \Phi_{e_1}^{G_1} \neq S$$

$$\text{The set } \left\{ \begin{aligned} c : c \Vdash (\exists x) \quad & \Phi_{e_0}^{G_0}(x) \downarrow \neq S(x) \vee \Phi_{e_0}^{G_0}(x) \uparrow \\ \vee \quad & \Phi_{e_1}^{G_1}(x) \downarrow \neq S(x) \vee \Phi_{e_1}^{G_1}(x) \uparrow \end{aligned} \right\} \text{ is dense}$$

IDEA: MAKE AN OVERAPPROXIMATION

"Can we find an extension for every instance of RT^1_2?"

Given a condition $c = (F_0, F_1, X)$, let $\psi(x, n)$ be the formula

$$(\forall B_0 \sqcup B_1 = \mathbb{N})(\exists i < 2)(\exists E_i \subseteq X \cap B_i)\Phi_{e_i}^{F_i \cup E_i}(x) \downarrow = n$$

$$\psi(x, n) \text{ is } \Sigma^{0,X}_1$$

Case 1: $\psi(x, n)$ holds

Letting $B_i = A_i$, there is an extension $d \leq c$ forcing

$$\Phi_{e_0}^{G_0}(x) \downarrow = n \vee \Phi_{e_1}^{G_1}(x) \downarrow = n$$

Case 2: $\psi(x, n)$ does not hold

$$(\exists B_0 \sqcup B_1 = \mathbb{N})(\forall i < 2)(\forall E_i \subseteq X \cap B_i)\Phi_{e_i}^{F_i \cup E_i}(x) \neq n$$

The condition $(F_0, F_1, X \cap B_i) \leq c$ forces

$$\Phi_{e_0}^{G_0}(x) \neq n \vee \Phi_{e_1}^{G_1}(x) \neq n$$

$$\mathcal{D} = \{(x, n) : \psi(x, n)\}$$

Σ_1 case

$(\exists x)(x, 1 - S(x)) \in \mathcal{D}$

Then $\exists d \leq c\ \exists i < 2$

$d \Vdash \Phi_{e_i}^{G_i}(x) \downarrow = 1 - S(x)$

Π_1 case

$(\exists x)(x, S(x)) \notin \mathcal{D}$

Then $\exists d \leq c\ \exists i < 2$

$d \Vdash \Phi_{e_i}^{G_i}(x) \neq S(x)$

Impossible case

$(\forall x)(x, 1 - S(x)) \notin \mathcal{D}$

$(\forall x)(x, S(x)) \in \mathcal{D}$

Then since $\mathcal{D}$ is X-c.e

$S \leq_T X$ ↯

RAMSEY'S THEOREM

Over n-tuples

$$\mathsf{RT}^n_k$$

Using k colors

RAMSEY'S THEOREM

Over n-tuples

$$\mathrm{RT}^{n}_{k,r}$$

Allows r colors

Using k colors

Thm (Wang)

A set is $\mathsf{RT}^n_{k,\ell}$-encodable iff it is computable for large ℓ

(whenever ℓ is at least the nth Schröder Number)

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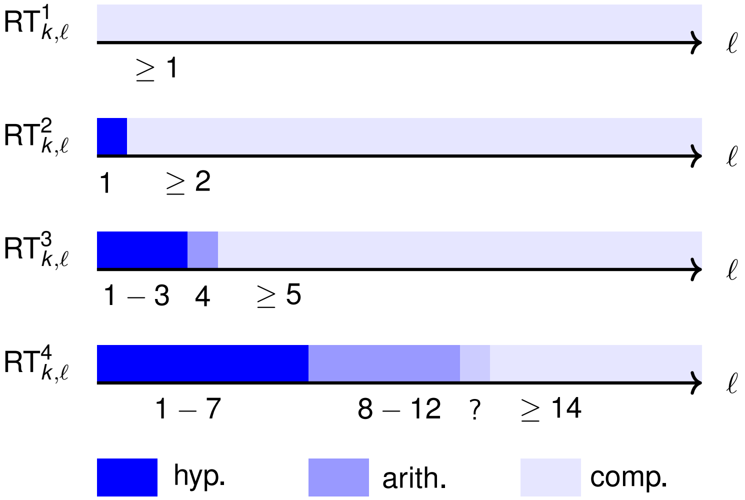
Thm (Cholak, P.)

A set is $\mathrm{RT}^n_{k,\ell}$-encodable iff it is arithmetic for medium ℓ

$\mathsf{RT}^n_{k,\ell}$-ENCODABLE SETS



The combinatorial features of RT^n_k reveal the computational features of RT^{n+1}_k

Open questions

Have we found the right framework?

Can variants of Mathias forcing
answer all Ramsey-type questions?

An infinite set C is $\vec{R}$-cohesive for some sets $R_0, R_1, \dots$ if for every i, either $C \subseteq^* R_i$ or $C \subseteq^* \overline{R}_i$.

COH : Every collection of sets has a cohesive set.

A coloring $f : [\omega]^2 \to 2$ is stable if $\lim_y f(x, y)$ exists for every x.

SRT^2_2 : Every stable coloring of pairs admits an infinite homogeneous set.

$$\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \leftrightarrow \mathsf{COH} \wedge \mathsf{SRT}^2_2$$

(Cholak, Jockusch and Slaman)

- Given $f : [\mathbb{N}]^2 \to 2$, define $\langle R_x : x \in \mathbb{N}\rangle$ by $R_x = \{y : f(x, y) = 1\}$
- By COH, there is an $\vec{R}$-cohesive set $C = \{x_0 < x_1 < \dots\}$
- $f : [C]^2 \to 2$ is stable

$$\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \leftrightarrow \mathsf{COH} \wedge \mathsf{SRT}^2_2$$

(Cholak, Jockusch and Slaman)

Thm (Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp, and Slaman)

$\mathsf{RCA}_0 \nvdash \mathsf{COH} \rightarrow \mathsf{SRT}^2_2$

Thm (Chong, Slaman and Yang)

$\mathsf{RCA}_0 \nvdash \mathsf{SRT}^2_2 \rightarrow \mathsf{COH}$

Using a non-standard model containing only low sets.

Does $\mathsf{SRT}^2_2 \models_c \mathsf{COH}$?

- Our analysis of SRT^2_2 is based on Mathias forcing
- Mathias forcing produces cohesive sets

Does $\mathsf{COH} \leq_c \mathsf{SRT}^2_2$?

COH admits a universal instance:

the primitive recursive sets

A set is p-cohesive if it is cohesive for the p.r. sets

Thm (Jockusch and Stephan)

A set is p-cohesive iff its jump is PA over $\emptyset'$

Thm (Jockusch and Stephan)

For every computable sequence of sets $\vec{R}$ and every p-cohesive set C, C computes an $\vec{R}$-cohesive set.

SRT^2_2 can be seen as a Δ^0_2 instance of
the pigeonhole principle

- Given a stable computable coloring $f : [\omega]^2 \to 2$
- Let $A = \{x : \lim_y f(x,y) = 1\}$
- Every infinite set $H \subseteq A$ or $H \subseteq \overline{A}$ computes an infinite f-homogeneous set.

Is there a set X such that
every infinite set $H \subseteq X$ or $H \subseteq \overline{X}$
has a jump of PA degree over $\emptyset'$?

Thm (Monin, P.)

Fix a non-Δ^0_2 set B. For every set X, there is an infinite set $H \subseteq X$ or $H \subseteq \overline{X}$ such that B is not $\Delta^{0,H}_2$.

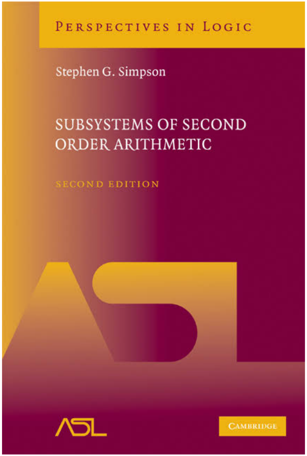
CONCLUSION

We have a minimalistic framework which answers accurately many questions about Ramsey's theorem.

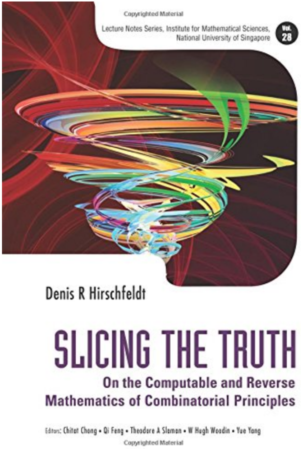
Ramsey-type problems compute through sparsity.

The computational properties of Ramsey-type problems are often immediate consequences of their combinatorics.

We understand what the Ramsey-type problems compute, but ignore what the jump of their solutions compute.



Subsystems of second-order arithmetic



Slicing the truth

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