Never underestimate pigeons

Ludovic PATEY

June 12, 2017

PIGEON-OWL PRINCIPLE

If you put infinitely many pigeons into finitely many owls,
one owl must contain infinitely many pigeons.

RAMSEY'S THEOREM

$[X]^n$ is the set of unordered n-tuples of elements of X

A k-coloring of $[X]^n$ is a map $f : [X]^n \to k$

A set $H \subseteq X$ is homogeneous for f if $|f([H]^n)| = 1$.

RT^n_k Every k-coloring of $[\mathbb{N}]^n$ admits an infinite homogeneous set.

PIGEONHOLE PRINCIPLE

RT^1_k Every k-partition of $\mathbb{N}$ admits an infinite part.

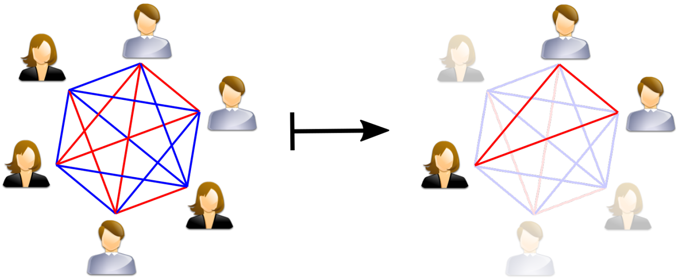
0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	....

$\longmapsto$

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	....

RAMSEY'S THEOREM FOR PAIRS

RT^2_k Every k-coloring of the infinite clique admits an infinite monochromatic subclique.



Motivations

REVERSE MATHEMATICS

Foundational program that seeks to determine the optimal axioms of ordinary mathematics.

REVERSE MATHEMATICS

Foundational program that seeks to determine the optimal axioms of ordinary mathematics.

$$\mathsf{RCA}_0 \vdash A \leftrightarrow T$$

in a very weak theory RCA_0
capturing computable mathematics

RCA_0

Robinson arithmetics

$$m+1 \neq 0$$
$$m+1 = n+1 \rightarrow m = n$$
$$\neg(m < 0)$$
$$m < n+1 \leftrightarrow (m < n \vee m = n)$$

$$m+0 = m$$
$$m+(n+1) = (m+n)+1$$
$$m \times 0 = 0$$
$$m \times (n+1) = (m \times n)+m$$

Σ^0_1 induction scheme

$$\varphi(0) \wedge \forall n(\varphi(n) \Rightarrow \varphi(n+1))$$
$$\Rightarrow \forall n \varphi(n)$$

where $\varphi(n)$ is Σ^0_1

Δ^0_1 comprehension scheme

$$\forall n(\varphi(n) \Leftrightarrow \psi(n))$$
$$\Rightarrow \exists X \forall n(n \in X \Leftrightarrow \varphi(n))$$

where $\varphi(n)$ is Σ^0_1 with free X, and ψ is Π^0_1.

Reverse mathematics

Mathematics are computationally
very structured

Almost every theorem is empirically equivalent to one among five big subsystems.

$$\Pi^1_1\mathrm{CA} \rightarrow \mathrm{ATR} \rightarrow \mathrm{ACA} \rightarrow \mathrm{WKL} \rightarrow \mathrm{RCA}_0$$

REVERSE MATHEMATICS

Mathematics are computationally very structured

Almost every theorem is empirically equivalent to one among five big subsystems.

$$\Pi^1_1\mathrm{CA} \to \mathrm{ATR} \to \mathrm{ACA} \to \mathrm{WKL} \to \mathrm{RCA}_0$$

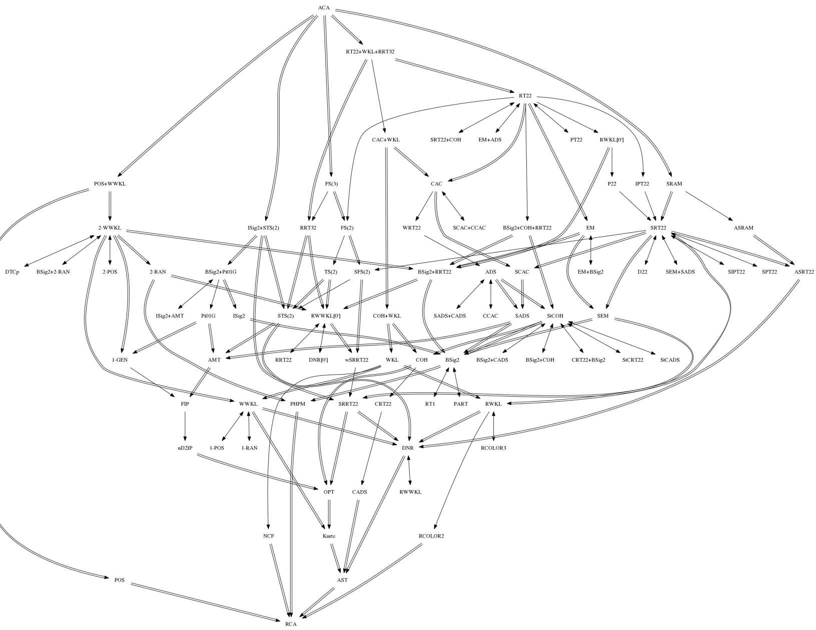
REVERSE MATHEMATICS

Mathematics are computationally very structured

$$\begin{tikzcd}
 & \Pi^1_1\mathrm{CA} \arrow[d] \\
 & \mathrm{ATR} \arrow[d] \\
 & \mathrm{ACA} \arrow[d] \arrow[dl] \\
 \mathrm{RT}^2_2 \arrow[dr] & \mathrm{WKL} \arrow[d] \\
 & \mathrm{RCA}_0
\end{tikzcd}$$

Almost every theorem is empirically equivalent to one among five big subsystems.

Except for Ramsey's theory...


ACA
RT22+WKL+RRT32
RT22
CAC+WKL
SRT22+COH
EM+ADS
PT22
RWKL[0']
POS+WWKL
FS(3)
CAC
P22
IPT22
SRAM
2-WWKL
ISig2+STS(2)
RRT32
FS(2)
WRT22
SCAC+CCAC
BSig2+COH+RRT22
EM
SRT22
ASRAM
DTCp
BSig2+2-RAN
2-POS
2-RAN
BSig2+Pi01G
TS(2)
SFS(2)
BSig2+RRT22
ADS
SCAC
EM+BSig2
D22
SEM+SADS
SIPT22
SPT22
ASRT22
ISig2+AMT
Pi01G
ISig2
STS(2)
RWWKL[0']
COH+WKL
SADS+CADS
CCAC
SADS
StCOH
SEM
I-GEN
AMT
RRT22
DNR[0']
wSRRT22
WKL
COH
BSig2
BSig2+CADS
BSig2+COH
CRT22+BSig2
StCRT22
StCADS
FIP
WWKL
PHPM
SRRT22
CRT22
RT1
PART
RWKL
nD2IP
I-POS
I-RAN
DNR
RCOLOR3
OPT
CADS
RWWKL
NCF
Kurtz
RCOLOR2
POS
AST
RCA

RT^1_2 and RT^2_2

The combinatorial features of RT_k^1 reveal the computational features of RT_k^2

An infinite set C is $\vec{R}$-cohesive for some sets $R_0, R_1, \dots$ if for every i, either $C \subseteq^* R_i$ or $C \subseteq^* \overline{R}_i$.

COH : Every collection of sets has a cohesive set.

COH is the bridge between RT^1_2 and RT^2_2

PROOF OF RT^2_2

- Let $f : [\omega]^2 \to 2$ be a coloring
- Define $\langle R_x : x \in \mathbb{N}\rangle$ by $R_x = \{y : f(x, y) = 1\}$
- By COH, there is an $\vec{R}$-cohesive set $C = \{x_0 < x_1 < \dots\}$
- Let $A = \{n : \lim_{t \in C} f(x_n, x_t)\}$
- By RT^1_2, there an infinite set $H \subseteq A$ or $H \subseteq \overline{A}$
- Compute a homogeneous set using C and H

PROOF OF RT^2_2

- Let $f : [\omega]^2 \to 2$ be a coloring
- Define $\langle R_x : x \in \mathbb{N} \rangle$ by $R_x = \{y : f(x, y) = 1\}$
- By COH, there is an $\vec{R}$-cohesive set $C = \{x_0 < x_1 < \dots\}$
- Let $A = \{n : \lim_{t \in C} f(x_n, x_t)\}$
- By RT^1_2, there an infinite set $H \subseteq A$ or $H \subseteq \overline{A}$
- Compute a homogeneous set using C and H

PROOF OF RT^2_2

- Let $f : [\omega]^2 \to 2$ be a coloring
- Define $\langle R_x : x \in \mathbb{N}\rangle$ by $R_x = \{y : f(x, y) = 1\}$
- By COH, there is an $\vec{R}$-cohesive set $C = \{x_0 < x_1 < \dots\}$
- Let $A = \{n : \lim_{t \in C} f(x_n, x_t)\}$
- By RT^1_2, there an infinite set $H \subseteq A$ or $H \subseteq \overline{A}$
- Compute a homogeneous set using C and H

PROOF OF RT^2_2

- Let $f : [\omega]^2 \to 2$ be a coloring
- Define $\langle R_x : x \in \mathbb{N}\rangle$ by $R_x = \{y : f(x,y) = 1\}$
- By COH, there is an $\vec{R}$-cohesive set $C = \{x_0 < x_1 < \dots\}$
- Let $A = \{n : \lim_{t \in C} f(x_n, x_t)\}$
- By RT^1_2, there an infinite set $H \subseteq A$ or $H \subseteq \overline{A}$
- Compute a homogeneous set using C and H

PROOF OF RT^2_2

- Let $f : [\omega]^2 \to 2$ be a coloring
- Define $\langle R_x : x \in \mathbb{N}\rangle$ by $R_x = \{y : f(x,y) = 1\}$
- By COH, there is an $\vec{R}$-cohesive set $C = \{x_0 < x_1 < \dots\}$
- Let $A = \{n : \lim_{t \in C} f(x_n, x_t)\}$
- By RT^1_2, there an infinite set $H \subseteq A$ or $H \subseteq \overline{A}$
- Compute a homogeneous set using C and H

PROOF OF RT^2_2

- Let $f : [\omega]^2 \to 2$ be a coloring
- Define $\langle R_x : x \in \mathbb{N}\rangle$ by $R_x = \{y : f(x, y) = 1\}$
- By COH, there is an $\vec{R}$-cohesive set $C = \{x_0 < x_1 < \dots\}$
- Let $A = \{n : \lim_{t \in C} f(x_n, x_t)\}$
- By RT^1_2, there an infinite set $H \subseteq A$ or $H \subseteq \overline{A}$
- Compute a homogeneous set using C and H

To analyse computable instances of RT^2_2,
we use computable instances of COH
and non-computable instances of RT^1_2

... and COH is computationally very weak

Avoidance

Let $\mathcal{C} \subseteq \omega^\omega$ be a closed set in the Baire space

Definition

A problem P avoids $\mathcal{C}$ if whenever $\mathcal{C}$ has no Z-computable member, for every Z-computable instance X of P, there is a solution Y such that $\mathcal{C}$ has no $Z \oplus Y$-computable member.

A problem P strongly avoids $\mathcal{C}$ if it avoids $\mathcal{C}$ for arbitrary instances of P.

EXAMPLES

- Avoiding a cone: $\mathcal{C}_X = \{X\}$

Thm (Seetapun)

RT^2_2 avoids cones

- Avoiding dominating functions: Given $f : \omega \to \omega$, $\mathcal{C}_f = \{g : g \geq f\}$

Thm (P.)

RT^2_2 avoids dominating one function

MORE EXAMPLES

- RT^2_2 avoids cones (Seetapun, 1995)
- RT^1_2 strongly avoids cones (Dzhafarov and J., 2009)
- WKL avoids dominating functions (J. and Soare, 1972)
- WKL avoids cones (J. and Soare, 1972)
- WKL does not avoid PA degrees (Solovay)
- WWKL avoids PA degrees (Kučera, 1985)
- RT^2_2 avoids PA degrees (Liu, 2012)
- RT^1_2 strongly avoids PA degrees (Liu, 2012)
- ...

If P avoids $\mathcal{C}$ but Q does not then

$$\mathsf{RCA}_0 \nvdash \mathsf{P} \rightarrow \mathsf{Q}$$

The combinatorial features of RT^1_k
reveal the computational features of RT^2_k

Thm (P.)

COH avoids every closed set

Thm (P.)

RT^2_k avoids a closed set iff RT^1_k strongly avoids it

RT^1_2 and RT^n_2

The computational analysis of RT^n_2 with more colors ressembles the analysis of RT^1_2

Fix a problem P.

A set S is P-encodable if there is an instance of P such that every solution computes S.

What sets can encode an instance of RT^n_k?

A function f is a modulus of a set S if every function dominating f computes S.

A set S is computably encodable if for every infinite set X, there is an infinite subset $Y \subseteq X$ computing S.

Thm (Solovay, Groszek and Slaman)

Given a set S, TFAE

- S is computably encodable
- S admits a modulus
- S is hyperarithmetic

Thm (Jockusch)

A set is RT^n_k-encodable for some $n \geq 2$ iff it is hyperarithmetic.

Thm (Jockusch)

A set is RT^n_k-encodable for some $n \geq 2$ iff it is hyperarithmetic.

Proof ($\Rightarrow$).

Let $g : [\omega]^n \to k$ be a coloring whose homogeneous sets compute S.

Since every infinite set has a homogeneous subset, S is computably encodable.

Thus S is hyperarithmetic. $\square$

Thm (Jockusch)

A set is RT^n_k-encodable for some $n \geq 2$ iff it is hyperarithmetic.

Proof ($\Leftarrow$).

Let S be hyperarithmetic with modulus μ_S.

Define $g : [\omega]^2 \to 2$ by $g(x, y) = 1$ iff $y > \mu_S(x)$.

Let $H = \{x_0 < x_1 < \dots\}$ be an infinite g-homogeneous set.

The function $p_H(n) = x_n$ dominates μ_S, hence computes S. $\square$

The encodability power of RT^n_k comes from the

sparsity

of its homogeneous sets.

What about RT^1_k?

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	....

Sparsity of red implies
non-sparsity of blue
and conversely.

Thm (Dzhafarov and Jockusch)

A set is RT^1_2-encodable iff it is computable.

Thm (Dzhafarov and Jockusch)

A set is RT^1_2-encodable iff it is computable.

Input : a set $S \not\leq_T \emptyset$ and a 2-partition $A_0 \sqcup A_1 = \mathbb{N}$

Output : an infinite set $G \subseteq A_i$ such that $S \not\leq_T G$

$$(F_0, F_1, X)$$

Initial segment ↗ ↖ Reservoir

- F_i is finite, X is infinite, $\max F_i < \min X$ (Mathias condition)
- $S \not\leq_T X$ (Weakness property)
- $F_i \subseteq A_i$ (Combinatorics)

Extension

$(E_0, E_1, Y) \leq (F_0, F_1, X)$

- $F_i \subseteq E_i$
- $Y \subseteq X$
- $E_i \setminus F_i \subseteq X$

Satisfaction

$\langle G_0, G_1 \rangle \in [F_0, F_1, X]$

- $F_i \subseteq G_i$
- $G_i \setminus F_i \subseteq X$

$$[E_0, E_1, Y] \subseteq [F_0, F_1, X]$$

$$\underbrace{(F_0, F_1, X)}_{\text{Condition}} \Vdash \underbrace{\varphi(G_0, G_1)}_{\text{Formula}}$$

$\varphi(G_0, G_1)$ holds for every $\langle G_0, G_1 \rangle \in [F_0, F_1, X]$

Input : a set $S \not\leq_T \emptyset$ and a 2-partition $A_0 \sqcup A_1 = \mathbb{N}$

Output : an infinite set $G \subseteq A_i$ such that $S \not\leq_T G$

Input : a set $S \not\leq_T \emptyset$ and a 2-partition $A_0 \sqcup A_1 = \mathbb{N}$

Output : an infinite set $G \subseteq A_i$ such that $S \not\leq_T G$

$$\Phi_{e_0}^{G_0} \neq S \vee \Phi_{e_1}^{G_1} \neq S$$

Input : a set $S \not\leq_T \emptyset$ and a 2-partition $A_0 \sqcup A_1 = \mathbb{N}$

Output : an infinite set $G \subseteq A_i$ such that $S \not\leq_T G$

$$\Phi_{e_0}^{G_0} \neq S \vee \Phi_{e_1}^{G_1} \neq S$$

The set $\left\{ \begin{array}{rl} c : c \Vdash (\exists x) & \Phi_{e_0}^{G_0}(x) \downarrow \neq S(x) \vee \Phi_{e_0}^{G_0}(x) \uparrow \\ \vee & \Phi_{e_1}^{G_1}(x) \downarrow \neq S(x) \vee \Phi_{e_1}^{G_1}(x) \uparrow \end{array} \right\}$ is dense

IDEA: MAKE AN OVERAPPROXIMATION

“Can we find an extension for every instance of RT^1_2?”

Given a condition $c = (F_0, F_1, X)$, let $\psi(x, n)$ be the formula

$$(\forall B_0 \sqcup B_1 = \mathbb{N})(\exists i < 2)(\exists E_i \subseteq X \cap B_i)\Phi_{e_i}^{F_i \cup E_i}(x) \downarrow = n$$

$$\psi(x, n) \text{ is } \Sigma^{0,X}_1$$

Case 1: $\psi(x, n)$ holds

Letting $B_i = A_i$, there is an extension $d \leq c$ forcing

$$\Phi_{e_0}^{G_0}(x)\downarrow = n \vee \Phi_{e_1}^{G_1}(x)\downarrow = n$$

Case 2: $\psi(x, n)$ does not hold

$$(\exists B_0 \sqcup B_1 = \mathbb{N})(\forall i < 2)(\forall E_i \subseteq X \cap B_i)\Phi_{e_i}^{F_i \cup E_i}(x) \neq n$$

The condition $(F_0, F_1, X \cap B_i) \leq c$ forces

$$\Phi_{e_0}^{G_0}(x) \neq n \vee \Phi_{e_1}^{G_1}(x) \neq n$$

$$\mathcal{D} = \{(x, n) : \psi(x, n)\}$$

Σ_1 case

$(\exists x)(x, 1 - S(x)) \in \mathcal{D}$

Then $\exists d \leq c\ \exists i < 2$

$d \Vdash \Phi^{G_i}_{e_i}(x) \downarrow= 1 - S(x)$

Π_1 case

$(\exists x)(x, S(x)) \notin \mathcal{D}$

Then $\exists d \leq c\ \exists i < 2$

$d \Vdash \Phi^{G_i}_{e_i}(x) \neq S(x)$

Impossible case

$(\forall x)(x, 1 - S(x)) \notin \mathcal{D}$

$(\forall x)(x, S(x)) \in \mathcal{D}$

Then since $\mathcal{D}$ is X-c.e

$S \leq_T X$ ↯

RAMSEY'S THEOREM

Over n-tuples

$$\mathrm{RT}^{n}_{k}$$

Using k colors

RAMSEY'S THEOREM

Over n-tuples

$$\mathsf{RT}^{n}_{k,r}$$

Allows r colors

Using k colors

Thm (Wang)

A set is $\mathrm{RT}^n_{k,\ell}$-encodable iff it is computable for large ℓ

(whenever ℓ is at least the *n*th Schröder Number)

Thm (Wang)

A set is $\mathsf{RT}^n_{k,\ell}$-encodable iff it is computable for large ℓ

(whenever ℓ is at least the nth Schröder Number)

Thm (Dorais, Dzhafarov, Hirst, Mileti, Shafer)

A set is $\mathsf{RT}^n_{k,\ell}$-encodable iff it is hyperarithmetic for small ℓ

(whenever $\ell < 2^{n-1}$)

Thm (Wang)

A set is $RT^n_{k,\ell}$-encodable iff it is computable for large ℓ

(whenever ℓ is at least the nth Schröder Number)

Thm (Dorais, Dzhafarov, Hirst, Mileti, Shafer)

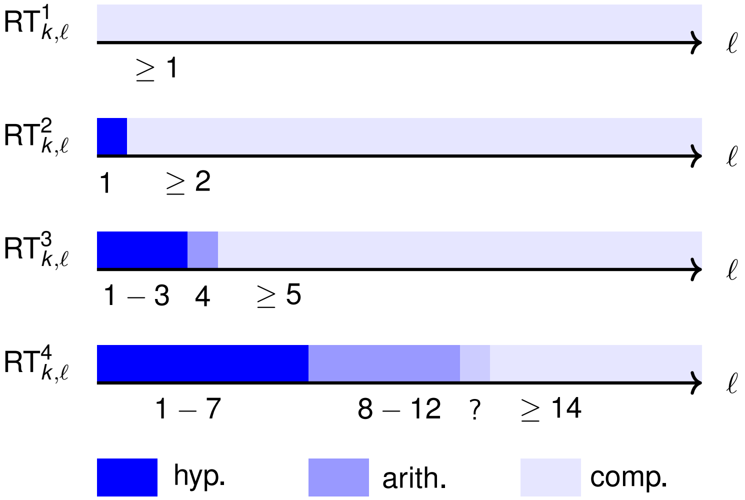
A set is $RT^n_{k,\ell}$-encodable iff it is hyperarithmetic for small ℓ

(whenever $\ell < 2^{n-1}$)

Thm (Cholak, P.)

A set is $RT^n_{k,\ell}$-encodable iff it is arithmetic for medium ℓ

RT$^n_{k,\ell}$-ENCODABLE SETS



Open questions

An infinite set C is $\vec{R}$-cohesive for some sets $R_0, R_1, \dots$ if for every i, either $C \subseteq^* R_i$ or $C \subseteq^* \overline{R}_i$.

COH : Every collection of sets has a cohesive set.

A coloring $f : [\omega]^2 \to 2$ is stable if $\lim_y f(x, y)$ exists for every x.

SRT^2_2 : Every stable coloring of pairs admits an infinite homogeneous set.

$$\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \leftrightarrow \mathsf{COH} \wedge \mathsf{SRT}^2_2$$

(Cholak, Jockusch and Slaman)

- Given $f : [\mathbb{N}]^2 \to 2$, define $\langle R_x : x \in \mathbb{N} \rangle$ by $R_x = \{y : f(x,y) = 1\}$
- By COH, there is an $\vec{R}$-cohesive set $C = \{x_0 < x_1 < \dots\}$
- $f : [C]^2 \to 2$ is stable

$$\mathrm{RCA}_0 \vdash \mathrm{RT}^2_2 \leftrightarrow \mathrm{COH} \wedge \mathrm{SRT}^2_2$$

(Cholak, Jockusch and Slaman)

Thm (Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp, and Slaman)

$\mathrm{RCA}_0 \nvdash \mathrm{COH} \rightarrow \mathrm{SRT}^2_2$

Thm (Chong, Slaman and Yang)

$\mathrm{RCA}_0 \nvdash \mathrm{SRT}^2_2 \rightarrow \mathrm{COH}$

Using a non-standard model containing only low sets.

Does SRT^2_2 imply COH over standard models?

- Our analysis of SRT^2_2 is based on Mathias forcing
- Mathias forcing produces cohesive sets

Does COH $\leq_c$ SRT^2_2?

COH admits a universal instance:

the primitive recursive sets

A set is p-cohesive if it is cohesive for the p.r. sets

Thm (Jockusch and Stephan)

A set is p-cohesive iff its jump is PA over $\emptyset'$

Thm (Jockusch and Stephan)

For every computable sequence of sets $\vec{R}$ and every p-cohesive set C, C computes an $\vec{R}$-cohesive set.

SRT^2_2 can be seen as a Δ^0_2 instance of the pigeonhole principle

- Given a stable computable coloring $f : [\omega]^2 \to 2$
- Let $A = \{x : \lim_y f(x,y) = 1\}$
- Every infinite set $H \subseteq A$ or $H \subseteq \overline{A}$ computes an infinite f-homogeneous set.

Is there a set X such that
every infinite set $H \subseteq X$ or $H \subseteq \overline{X}$
has a jump of PA degree over $\emptyset'$?

Thm (Monin, P.)

Fix a non-Δ^0_2 set B. For every set X, there is an infinite set $H \subseteq X$ or $H \subseteq \overline{X}$ such that B is not $\Delta^{0,H}_2$.

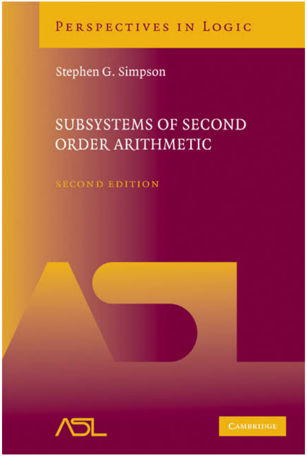
CONCLUSION

Understanding Ramsey's theorem requires understanding the pigeonhole principle.

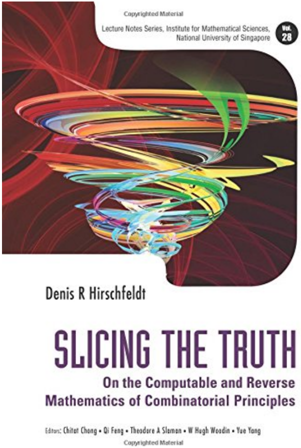
Ramsey-type problems compute through sparsity.

The computational properties of Ramsey-type problems are often immediate consequences of their combinatorics.

We understand what the Ramsey-type problems compute, but ignore what the jump of their solutions compute.



Subsystems of second-order arithmetic



Slicing the truth

References

Peter A. Cholak, Carl G. Jockusch, and Theodore A. Slaman.
On the strength of Ramsey's theorem for pairs.
Journal of Symbolic Logic, 66(01):1–55, 2001.

Carl G. Jockusch.
Ramsey's theorem and recursion theory.
Journal of Symbolic Logic, 37(2):268–280, 1972.

Ludovic Patey.
The reverse mathematics of Ramsey-type theorems.
PhD thesis, Université Paris Diderot, 2016.

Wei Wang.
Some logically weak Ramseyan theorems.
Advances in Mathematics, 261:1–25, 2014.