Partial orders and immunity in reverse mathematics

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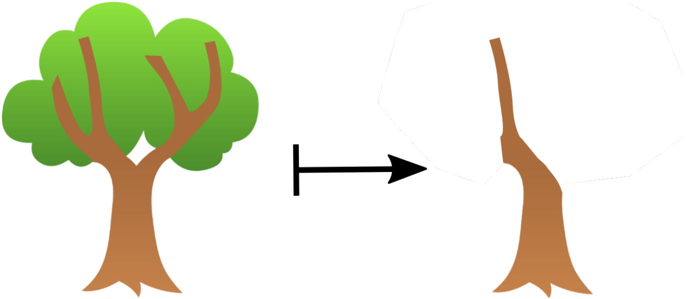
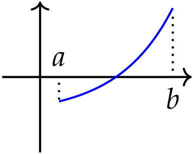
Many theorems can be seen as problems.

König's lemma
Every infinite, finitely branching tree admits an infinite path.



Some theorems are more effective than others.

Intermediate value theorem
For every continuous function f over an interval $[a, b]$ such that $f(a) \cdot f(b) < 0$, there is a real $x \in [a, b]$ such that $f(x) = 0$.



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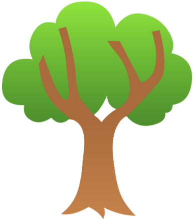

REVERSE MATHEMATICS

Q is at least as hard as P if

$$\mathsf{RCA}_0 \vdash \mathsf{Q} \to \mathsf{P}$$

in a very weak theory RCA_0
capturing computable mathematics

(Harvey Friedman, 1974)

Turing ideal $\mathcal{M}$

- $(\forall X \in \mathcal{M})(\forall Y \leq_T X)[Y \in \mathcal{M}]$
- $(\forall X, Y \in \mathcal{M})[X \oplus Y \in \mathcal{M}]$

Examples

- $\{X : X \text{ is computable }\}$
- $\{X : X \leq_T A \wedge X \leq_T B\}$ for some sets A and B

Let $\mathcal{M}$ be a Turing ideal and P, Q be problems.

Satisfaction

$$\mathcal{M} \models \mathsf{P}$$

if every P-instance in $\mathcal{M}$ has a solution in $\mathcal{M}$.

Computable entailment

$$\mathsf{P} \models_c \mathsf{Q}$$

if every Turing ideal satisfying P satisfies Q.

Fix two problems P and Q.

How to prove that $\mathsf{P} \not\models_c \mathsf{Q}$?

Build a Turing ideal $\mathcal{M}$ such that

- $\mathcal{M} \models \mathsf{P}$
- $\mathcal{M} \not\models \mathsf{Q}$

PROVING $\mathsf{P} \not\models_c \mathsf{Q}$

Pick a Q-instance I with no I-computable solution

Start with $\mathcal{M}_0 = \{Z : Z \leq_T I\}$

Given a Turing ideal $\mathcal{M}_n = \{Z : Z \leq_T U\}$ for some set U,

PROVING $\mathsf{P} \not\models_c \mathsf{Q}$

Pick a Q-instance I with no I-computable solution

Start with $\mathcal{M}_0 = \{Z : Z \leq_T I\}$

Given a Turing ideal $\mathcal{M}_n = \{Z : Z \leq_T U\}$ for some set U,

1. pick some P-instance $X \in \mathcal{M}_n$

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Start with $\mathcal{M}_0 = \{Z : Z \leq_T I\}$

Given a Turing ideal $\mathcal{M}_n = \{Z : Z \leq_T U\}$ for some set U,

1. pick some P-instance $X \in \mathcal{M}_n$
2. choose a solution Y to X
3. let $\mathcal{M}_{n+1} = \{Z : Z \leq_T Y \oplus U\}$

Beware, while adding sets to $\mathcal{M}$,
we may add a solution to the Q-instance!

A weakness property is a collection of sets closed downwards under the Turing reducibility.

Examples

- $\{X : X \text{ is low}\}$
- $\{X : A \not\leq_T X\}$ for some set A
- $\{X : X \text{ is hyperimmune-free}\}$

Fix a weakness property $\mathcal{W}$.

A problem P preserves $\mathcal{W}$ if for every $Z \in \mathcal{W}$,
every Z-computable P-instance X
has a solution Y such that $Y \oplus Z \in \mathcal{W}$

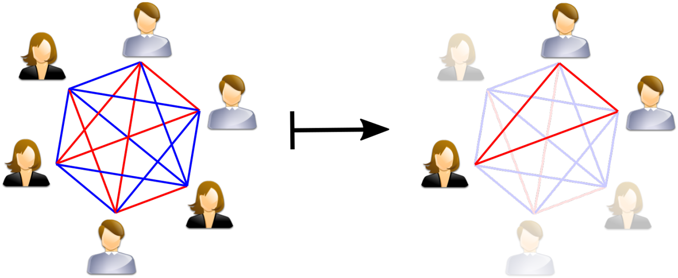
Lemma
If P *preserves* $\mathcal{W}$ *but* Q *does not, then* $\mathsf{P} \not\models_c \mathsf{Q}$

Find the right weakness properties

- $\mathsf{WKL} \not\models_c \mathsf{ACA}$ (cone avoidance)
- $\mathsf{RT}^2_2 \not\models_c \mathsf{ACA}$ (cone avoidance)
- $\mathsf{EM} \not\models_c \mathsf{RT}^2_2$ (2 hyperimmunities)
- $\mathsf{EM} \not\models_c \mathsf{TS}^2$ (ω hyperimmunities)
- $\mathsf{TS}^2 \not\models_c \mathsf{RT}^2_2$ (2 hyperimmunities)
- $\mathsf{RT}^2_2 \not\models_c \mathsf{TT}^2_2$ (fairness property)
- $\mathsf{RT}^2_2 \not\models_c \mathsf{WWKL}$ (c.b-enum avoidance)
- ...

RAMSEY'S THEOREM

RT^n_k Every k-coloring of $[\mathbb{N}]^n$ admits an infinite homogeneous set.



CAC

Every infinite partial order admits
an infinite chain or antichain.

Let $\mathcal{L} = (\omega, \leq_{\mathcal{L}})$ be a partial order.

$$f(\{x, y\}) = \begin{cases} 0 & \text{if } x <_{\mathcal{L}} y \vee y <_{\mathcal{L}} x \\ 2 & \text{if } x \mid_{\mathcal{L}} y \end{cases}$$

Any infinite f-homogeneous set
is a chain or an antichain.

$$\mathsf{CAC} \not\models_c \mathsf{RT}^2_2$$

(Hirschfeldt and Shore)

A function f is DNC if $(\forall e)[f(e) \neq \Phi_e(e)]$

Let $\mathcal{W}_{DNC} = \{Z : Z \text{ does not compute a DNC function}\}$

CAC preserves $\mathcal{W}_{DNC}$ but RT^2_2 does not

$\mathsf{CAC} \not\models_c \mathsf{RT}^2_2$

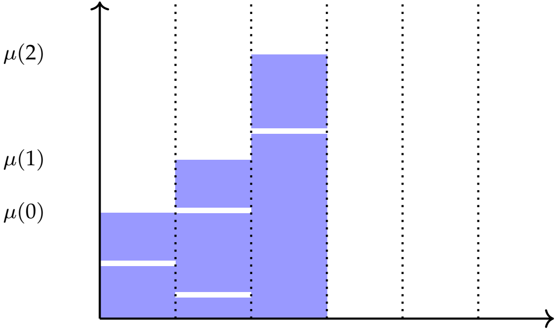
(Hirschfeldt and Shore)

A *k*-enum of X is a sequence $F_0 < F_1 < \dots$ of sets such that $|F_i| = k$ and $F_i \cap X \neq \emptyset$ for every $i \in \mathbb{N}$

Let $\mathcal{W}^X_{Enum} = \{Z : Z \text{ does not compute a k-enum of } X\}$

CAC preserves $\mathcal{W}^X_{Enum}$ for every X, but RT^2_2 does not

There is an X with no computable k-enum such that every DNC function computes an infinite subset of X.



ADS

Every infinite linear order admits an infinite ascending or descending sequence.

Let $\mathcal{L} = (\omega, \leq_{\mathcal{L}})$ be a linear order.

$x \leq_{\mathcal{P}} y$ iff $x <_{\mathbb{N}} y \wedge x \leq_{\mathcal{L}} y$

Any infinite chain or antichain for $\mathcal{P}$ is an ascending or descending sequence for $\mathcal{L}$.

ADS $\not\models_c$ CAC

(Lerman, Solomon and Towsner)

$\varphi(U, V)$ is essential if $(\forall x)(\exists R > x)(\forall y)(\exists S > y)\varphi(R, S)$

X, Y are dependently Z-hyperimmune if for every essential $\Sigma^{0,Z}_1$ formula $\varphi(U, V)$, $\varphi(R, S)$ holds for some $R \subseteq \overline{X}$ and $S \subseteq \overline{Y}$

Let $\mathcal{W}^{X,Y}_{DH} = \{Z : X, Y$ are dependently Z-hyperimmune$\}$

ADS preserves $\mathcal{W}^{X,Y}_{DH}$ for every X, Y, but CAC does not

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