Coloring trees in reverse mathematics

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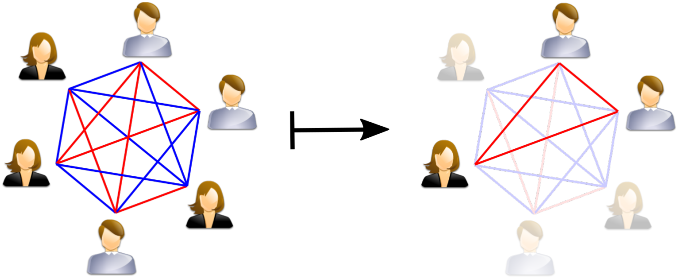
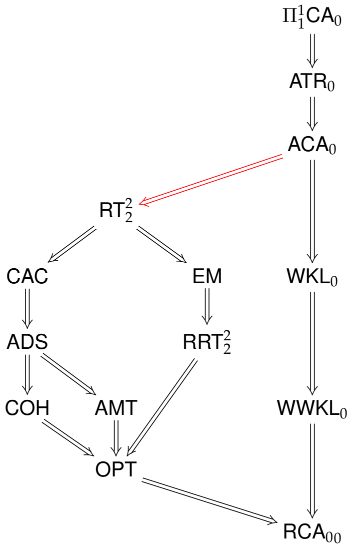
Joint work with Damir Dzhafarov - Oct 22, 2016

RAMSEY'S THEOREM

RT^n_k Every k-coloring of $[\mathbb{N}]^n$ admits an infinite homogeneous set.




$\Pi^1_1CA_0$
ATR_0
ACA_0
RT^2_2
CAC
EM
WKL_0
ADS
RRT^2_2
COH
AMT
$WWKL_0$
OPT
RCA_{00}

Is there a natural principle P such that

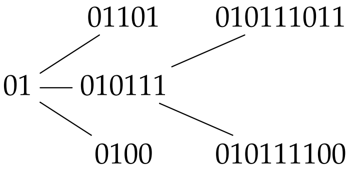
$$\mathsf{RT}^2_2 < \mathsf{P} < \mathsf{ACA}_0\,?$$

$$\mathsf{RT}^2_2 < \mathsf{RT}^2_2 \wedge \mathsf{WKL} < \mathsf{ACA}_0$$

Trees

$$T = \{01, 01101, 010111, 0100, 010111011, 010111100\}$$

Tree structure



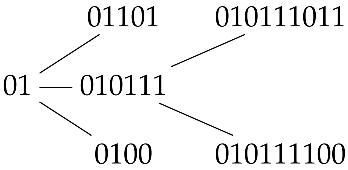
Terminology

- 010111100 is a successor of 010111 in T
- 01101 is a leaf of T

COMPARABLE NODES

$$T = \{01, 01101, 010111, 0100, 010111011, 010111100\}$$

Tree structure



Terminology

- $[T]^n$ set of n-tuples of comparable nodes in T.
- $\{01, 010111011\} \in [T]^2$
- $\{01101, 010111011\} \notin [T]^2$

THE TREE THEOREM

TT^n_k Every k-coloring of $[2^{<\omega}]^n$ admits a homogeneous set $T \cong 2^{<\omega}$.

(Chubb, Hirst and McNicholl)

$T \cong 2^{<\omega}$: T is isomorphic to $2^{<\omega}$

$$\mathsf{RCA}_0 \vdash \mathsf{TT}^n_k \to \mathsf{RT}^n_k$$

(Chubb, Hirst and McNicholl)

Let f be a k-coloring of $[\omega]^n \to k$

$$g(\sigma_1, \ldots, \sigma_k) = f(|\sigma_1|, \ldots, |\sigma_k|)$$

The strength of TT^1

$$\mathsf{RCA}_0 \vdash \mathsf{TT}^1_k$$

(Chubb, Hirst and McNicholl)

$$\begin{array}{c} \mathsf{I}\Sigma^0_2 \\ \Big\Downarrow \\ \mathsf{B}\Sigma^0_2 \\ \Downarrow \\ \mathsf{I}\Sigma^0_1 \end{array}$$

$$
\begin{array}{c}
\mathsf{I}\Sigma^0_2 \\
\Downarrow \\
\mathsf{B}\Sigma^0_2 = \mathsf{RT}^1 \\
\Downarrow \\
\mathsf{I}\Sigma^0_1
\end{array}
$$

(Cholak, Jockusch and Slaman)

$$
\begin{tikzcd}
\mathsf{I}\Sigma^0_2 \arrow[dd, Rightarrow] \arrow[dr] & \\
& \mathsf{TT}^1 \arrow[d] \\
\mathsf{B}\Sigma^0_2 \arrow[d, Rightarrow] \arrow[r, equal] & \mathsf{RT}^1 \\
\mathsf{I}\Sigma^0_1 &
\end{tikzcd}
$$

(Chubb, Hirst and McNicholl)

$$\begin{tikzcd}
\mathsf{I}\Sigma^0_2 \arrow[dr] \arrow[dd, Rightarrow] & \\
& \mathsf{TT}^1 \arrow[d, Rightarrow] \\
\mathsf{B}\Sigma^0_2 \arrow[r, equal] \arrow[d, Rightarrow] & \mathsf{RT}^1 \\
\mathsf{I}\Sigma^0_1 &
\end{tikzcd}$$

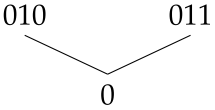
(Corduan, Groszek and Mileti)

Lemma (Corduan, Groszek and Mileti)

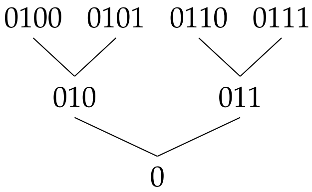
There is a computable function $f : \mathbb{N} \to [2^{<\omega}]^2 \to 2$ *such that for every* n *and every* $e < n$, Φ_e *is not a solution to* $f(n)$.

0

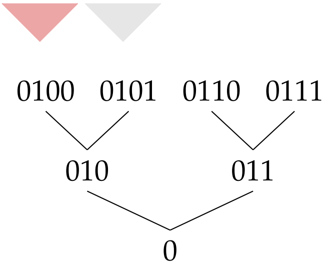
Φ_0 Φ_1



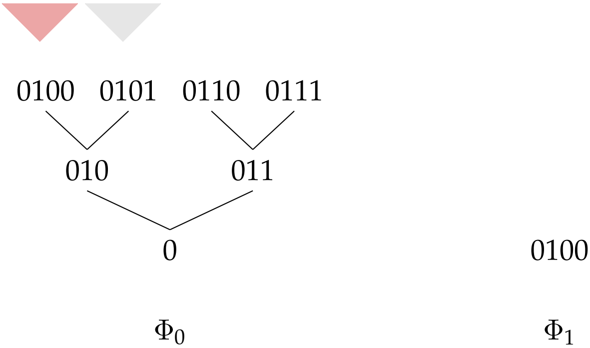
Φ_0 Φ_1

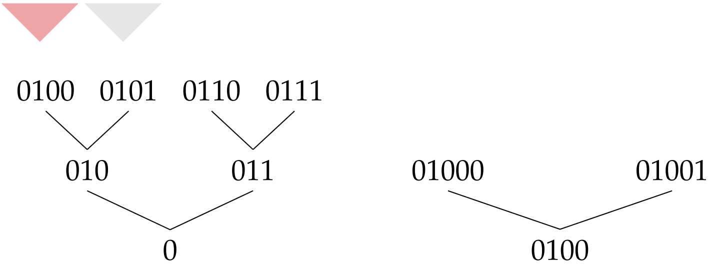


Φ_0 Φ_1

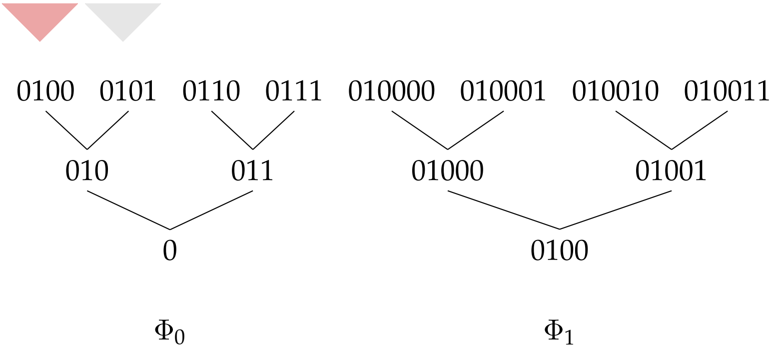


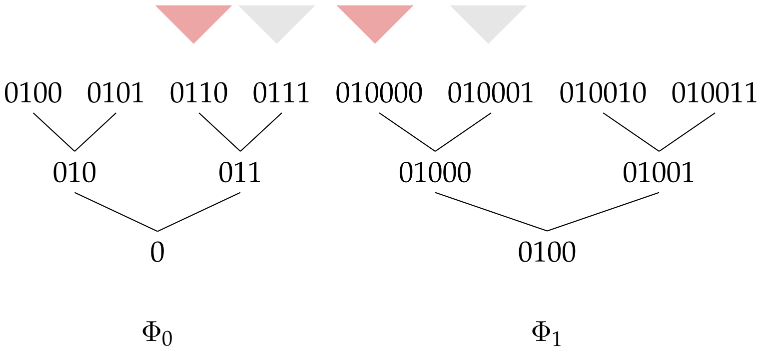
Φ_0 Φ_1


0100 0101 0110 0111
010 011
0
0100
Φ_0
Φ_1



Φ_0 Φ_1


0100 0101 0110 0111 010000 010001 010010 010011
010 011 01000 01001
0 0100
Φ0 Φ1


0100
0101
0110
0111
010000
010001
010010
010011
010
011
01000
01001
0
0100
Φ_0
Φ_1

The strength of TT^n

$$\begin{array}{c} \mathsf{ACA}' \\ \Downarrow \\ \mathsf{ACA} \\ \Downarrow \\ \mathsf{RCA}_0 \end{array}$$

$$\begin{array}{ccc}
\mathsf{ACA}' & & \\
\Downarrow & & \\
\mathsf{ACA} & = & \mathsf{RT}^n \\
\Downarrow & & \\
\mathsf{RCA}_0 & & (n \geq 3)
\end{array}$$

(Jockusch and Simpson)

$$\begin{tikzcd}
\mathsf{ACA}' \arrow[r, equal] \arrow[d, Rightarrow] & \mathsf{RT} \arrow[d, Rightarrow] \\
\mathsf{ACA} \arrow[r, equal] \arrow[d, Rightarrow] & \mathsf{RT}^n \\
\mathsf{RCA}_0 &
\end{tikzcd}
\qquad (n \geq 3)$$

(Friedman, McAloon, and Simpson)

$$\begin{tikzcd}
\mathsf{ACA}' \arrow[r, equal] \arrow[d, Rightarrow] & \mathsf{RT} \arrow[d, Rightarrow] \\
\mathsf{ACA} \arrow[r, equal] \arrow[dd, Rightarrow] & \mathsf{RT}^n \arrow[d] \\
 & \mathsf{RT}^2_k \arrow[dl, Rightarrow] \\
\mathsf{RCA}_0 &
\end{tikzcd}$$

$(n \geq 3)$

(Jockusch)

$$\begin{tikzcd}
\mathsf{ACA}' \arrow[r, equal] \arrow[d, Rightarrow] & \mathsf{RT} \arrow[d, Rightarrow] \\
\mathsf{ACA} \arrow[r, equal] \arrow[dd, Rightarrow] & \mathsf{RT}^n \arrow[d, Rightarrow, blue] \\
 & \mathsf{RT}^2_k \arrow[dl, Rightarrow] \\
\mathsf{RCA}_0 &
\end{tikzcd} \qquad (n \geq 3)$$

(Seetapun and Slaman)

$$
\begin{tikzcd}
\mathsf{ACA}' \arrow[r, equal] \arrow[d, Rightarrow] & \mathsf{RT} \arrow[d, Rightarrow] & \\
\mathsf{ACA} \arrow[r, equal] \arrow[ddd, Rightarrow] & \mathsf{RT}^n \arrow[r, equal, blue] \arrow[dd, Rightarrow] & \mathsf{TT}^n \\
& & \\
& \mathsf{RT}^2_k \arrow[dl, Rightarrow] & \\
\mathsf{RCA}_0 & & (n \geq 3)
\end{tikzcd}
$$

(Corduan, Groszek and McNicholl)

$$
\begin{tikzcd}
\mathsf{ACA}' \arrow[r, equal] \arrow[d, Rightarrow] & \mathsf{RT} \arrow[r, equal] \arrow[d, Rightarrow] & \mathsf{TT} \arrow[d, Rightarrow] \\
\mathsf{ACA} \arrow[r, equal] \arrow[dd, Rightarrow] & \mathsf{RT}^n \arrow[r, equal] \arrow[d, Rightarrow] & \mathsf{TT}^n \\
& \mathsf{RT}^2_k \arrow[dl, Rightarrow] & \\
\mathsf{RCA}_0 & & (n \geq 3)
\end{tikzcd}
$$

(Anderson and Hirst)

$$
\begin{tikzcd}
\mathsf{ACA}' \arrow[r, equal] \arrow[d, Rightarrow] & \mathsf{RT} \arrow[r, equal] \arrow[d, Rightarrow] & \mathsf{TT} \arrow[d, Rightarrow] \\
\mathsf{ACA} \arrow[r, equal] \arrow[dddd, Rightarrow] & \mathsf{RT}^n \arrow[r, equal] \arrow[dd, Rightarrow] & \mathsf{TT}^n \arrow[d] \\
 & & \mathsf{TT}^2_k \arrow[dl, Rightarrow, blue] \\
 & \mathsf{RT}^2_k \arrow[dl, Rightarrow] & \\
\mathsf{RCA}_0 & & (n \geq 3)
\end{tikzcd}
$$

(P.)

$$
\begin{tikzcd}
\mathsf{ACA}' \arrow[r, equal] \arrow[d, Rightarrow] & \mathsf{RT} \arrow[r, equal] \arrow[d, Rightarrow] & \mathsf{TT} \arrow[d, Rightarrow] \\
\mathsf{ACA} \arrow[r, equal] \arrow[ddd, Rightarrow] & \mathsf{RT}^n \arrow[r, equal] \arrow[dd, Rightarrow] & \mathsf{TT}^n \arrow[d, Rightarrow, blue] \\
 & & \mathsf{TT}^2_k \arrow[dl, Rightarrow] \\
 & \mathsf{RT}^2_k \arrow[dl, Rightarrow] & \\
\mathsf{RCA}_0 & & (n \geq 3)
\end{tikzcd}
$$

(Dzhafarov and P.)

Cone avoidance of TT^2

$f : [\omega]^2 \to k$ stable on $H \subseteq \omega$

$$(\forall x \in H)(\exists c < k)(\forall^\infty y \in H)$$
$$f(x, y) = c$$

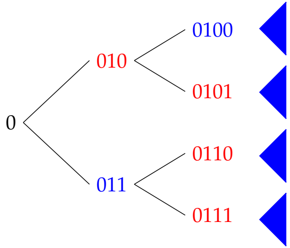
$$y \mapsto f(0, y)$$

0 1 2 3 4 5 6 7 8 9 10 11 $\cdots$

$f : [2^{<\omega}]^2 \to k$ stable on $H \subseteq 2^{<\omega}$

$$(\forall \sigma \in H)(\exists c < k)(\forall^\infty \tau \in H)$$
$$\sigma \prec \tau \to f(\sigma, \tau) = c$$

$$\sigma \mapsto f(0, \sigma)$$



Ramsey's theorem for pairs

$\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \leftrightarrow \Delta^0_2(\mathsf{RT}^1_2) \wedge \mathsf{CRT}^2_2$

$\Delta^0_2(\mathsf{RT}^1_2)$: Every Δ^0_2 instance of RT^1_2 has a solution.

CRT^2_2: Every instance of RT^2_2 has an infinite stable set H.

The tree theorem for pairs

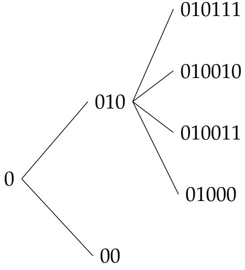
$\mathsf{RCA}_0 \vdash \mathsf{TT}^2_2 \leftrightarrow \Delta^0_2(\mathsf{TT}^1_2) \wedge \mathsf{CTT}^2_2$

$\Delta^0_2(\mathsf{TT}^1_2)$: Every Δ^0_2 instance of TT^1_2 has a solution.

CTT^2_2: Every instance of TT^2_2 has an infinite stable set H.

CONE AVOIDANCE OF CTT^2_2

A set $T \subseteq 2^{<\omega}$ is h-branching if every non-leaf $\sigma \in T$ at level n has exactly $h(n)$ successors.

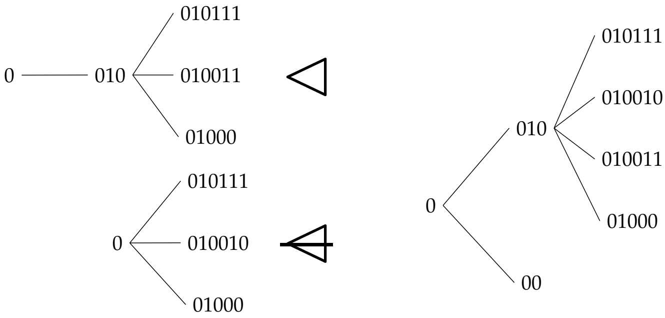


$$h(n) = 2n$$

CONE AVOIDANCE OF CTT^2_2

$$S \lhd T$$

"$S \subseteq T$ and $(\forall \tau \in S)(\forall \sigma \in T)[\sigma \prec \tau \to \sigma \in S]$"



CONE AVOIDANCE OF CTT^2_2

Let T be a finite $(g+h)$-branching set whose leaves are blue or red. Then

there is a g-branching $S \lhd T$ whose leaves are blue

or

there is an h-branching $S \lhd T$ whose leaves are red

CONE AVOIDANCE OF CTT^2_2

Choose h so that in any finite h-branching set T, there is a valid 2-branching $S \lhd T$.

Is there a finite h-branching tree T such that every 2-branching $S \lhd T$ contains a Φ-splitting?

If yes, then there is a valid $S \lhd T$ which has a Φ-splitting.

If no, then there is an infinite 2-branching subtree with no Φ-splitting. Work within it.

Strong cone avoidance of RT^1_2

Fix a set $A \subseteq \omega$ and an ω-model $\mathcal{M} \models \mathsf{WKL}$ with $\emptyset' \notin \mathcal{M}$.

- Case 1: $A \cap X$ is X-hyperimmune for some infinite set $X \in \mathcal{M}$. Then $\overline{A}$ has an infinite incomplete subset.
- Case 2: $A \cap X$ is X-hyperimmune for no infinite set $X \in \mathcal{M}$. Then A has an infinite incomplete subset.

(Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp and Slaman)

Strong cone avoidance of TT^1_2

Fix a set $A \subseteq 2^{<\omega}$ and an ω-model $\mathcal{M} \models \mathsf{WKL}$ with $\emptyset' \notin \mathcal{M}$.

- Case 1: $A \cap X$ is densely X-hyperimmune for some set $X \in \mathcal{M}$ such that $X \cong 2^{<\omega}$. Then $\overline{A}$ has an incomplete subset $H \cong 2^{<\omega}$.
- Case 2: $A \cap X$ is densely X-hyperimmune for no set $X \in \mathcal{M}$ such that $X \cong 2^{<\omega}$. Then A has an incomplete subset $H \cong 2^{<\omega}$.

(Dzhafarov and P.)

SUMMARY

$$
\begin{tikzcd}
\mathsf{ACA}' \arrow[r, equal] \arrow[d, Rightarrow] & \mathsf{RT} \arrow[r, equal] \arrow[d, Rightarrow] & \mathsf{TT} \arrow[d, Rightarrow] \\
\mathsf{ACA} \arrow[r, equal] \arrow[ddd, Rightarrow] & \mathsf{RT}^n \arrow[r, equal] \arrow[dd, Rightarrow] & \mathsf{TT}^n \arrow[d, Rightarrow] \\
 & & \mathsf{TT}^2_k \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \\
 & \mathsf{RT}^2_k \arrow[d, Rightarrow] & \mathsf{TT}^1 \arrow[dl, Rightarrow] \\
\mathsf{B}\Sigma^0_2 \arrow[r, equal] \arrow[d, Rightarrow] & \mathsf{RT}^1 & \\
\mathsf{RCA}_0 & & (n \geq 3)
\end{tikzcd}
$$

REFERENCES

Bernard A. Anderson and Jeffry L. Hirst.
Partitions of trees and ACA_0'.
Arch. Math. Logic, 48(3-4):227–230, 2009.

Jennifer Chubb, Jeffry L. Hirst, and Timothy H. McNicholl.
Reverse mathematics, computability, and partitions of trees.
Journal of Symbolic Logic, 74(01):201–215, 2009.

Jared Corduan, Marcia J. Groszek, and Joseph R. Mileti.
Reverse mathematics and Ramsey's property for trees.
Journal of Symbolic Logic, 75(03):945–954, 2010.

Damir D Dzhafarov and Ludovic Patey.
Coloring trees in reverse mathematics.
To appear. Available at `https://arxiv.org/abs/1609.02627`.

Ludovic Patey.
The strength of the tree theorem for pairs in reverse mathematics.
Journal of Symbolic Logic, 2015.
To appear. Available at `http://arxiv.org/abs/1505.01057`.