Iterative forcing and preservation of hyperimmunity

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THEOREMS AS PROBLEMS

Examples:

- (König's lemma)
  Every infinite, finitely branching tree has an infinite path.
- (Ramsey's theorem)
  Every k-coloring has an infinite monochromatic subset.
- (The atomic model theorem)
  Every complete atomic theory has an atomic model.
- ...

THEOREMS AS PROBLEMS

Many theorems P are of the form

$$(\forall X)[\Phi(X) \to (\exists Y)\Psi(X, Y)]$$

where Φ and Ψ are arithmetic formulas.

We may think of P as a class of problems.

- An X such that $\Phi(X)$ holds is an instance.
- A Y such that $\Psi(X, Y)$ holds is a solution to X.

TURING IDEALS

A Turing ideal is a collection of sets $\mathcal{M}$ closed under

- the Turing reduction: $(\forall X \in \mathcal{M})(\forall Y \leq_T X)[Y \in \mathcal{M}]$
- the effective join: $(\forall X, Y \in \mathcal{M})[X \oplus Y \in \mathcal{M}]$

Example:

- $\{X : X \text{ is computable } \}$
- $\{X : X \leq_T A \wedge X \leq_T B\}$ for some sets A and B

COMPARE THEOREMS

A Turing ideal $\mathcal{M}$ satisfies a theorem P (written $\mathcal{M} \models \mathsf{P}$) if every P-instance in $\mathcal{M}$ has a solution in $\mathcal{M}$.

A theorem P entails a theorem Q (written $\mathsf{P} \vdash \mathsf{Q}$) if every Turing ideal satisfying P satisfies Q.

SEPARATING THEOREMS

Fix two theorems P and Q.

How to prove that $\mathsf{P} \not\vdash \mathsf{Q}$?

Build a Turing ideal $\mathcal{M}$ such that

- $\mathcal{M} \models \mathsf{P}$
- $\mathcal{M} \not\models \mathsf{Q}$

SEPARATING THEOREMS

Pick a Q-instance I with no I-computable solution.

Start with $\mathcal{M}_0 = \{Z : Z \leq_T I\}$.

Given a Turing ideal $\mathcal{M}_n = \{Z : Z \leq_T U\}$ for some set U,

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1. pick some P-instance $X \in \mathcal{M}_n$

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Given a Turing ideal $\mathcal{M}_n = \{Z : Z \leq_T U\}$ for some set U,

1. pick some P-instance $X \in \mathcal{M}_n$
2. choose a solution Y to X
3. let $\mathcal{M}_{n+1} = \{Z : Z \leq_T Y \oplus U\}$.

SEPARATING THEOREMS

Beware, while adding sets to $\mathcal{M}$,
we may add a solution to the Q-instance!

SEPARATING THEOREMS

A weakness property is a collection of sets closed downwards under the Turing reducibility.

Examples

- $\{X : X \text{ is low}\}$
- $\{X : A \not\leq_T X\}$ for some set A
- $\{X : X \text{ is hyperimmune-free}\}$

SEPARATING THEOREMS

Fix a property $\mathcal{P}$.

A statement P preserves $\mathcal{P}$ if for every $Z \in \mathcal{P}$, every Z-computable P-instance X has a solution Y such that $Y \oplus Z \in \mathcal{P}$

Lemma

If P preserves $\mathcal{P}$ but Q does not, then $\mathsf{P} \nvdash \mathsf{Q}$

SEPARATING THEOREMS

Let V witness that Q does not preserve $\mathcal{P}$.
Start with $\mathcal{M}_0 = \{Z : Z \leq_T V\} \subseteq \mathcal{P}$

Given a Turing ideal $\mathcal{M}_n = \{Z : Z \leq_T U\}$ for some set $U \in \mathcal{P}$,

1. pick some P-instance $X \in \mathcal{M}_n$
2. choose a solution Y to X such that $Y \oplus U \in \mathcal{P}$
3. let $\mathcal{M}_{n+1} = \{Z : Z \leq_T U \oplus Y\} \subseteq \mathcal{P}$.

AN EXAMPLE

Given a sequence of non-c.e. sets $A_0, A_1, \ldots$

$$\mathcal{P}_{\vec{A}} = \{Z : \text{ the A's are not Z-c.e.}\}$$

Theorem (Wei Wang)

- *For every countable sequence of non-c.e. sets $A_0, A_1, \ldots$, weak König's lemma, the Erdős-Moser theorem, and cohesiveness preserve $\mathcal{P}_{\vec{A}}$.*
- *There is a countable sequence of non-c.e. sets $A_0, A_1, \ldots$ such that the thin set theorem for pairs does not preserve $\mathcal{P}_{\vec{A}}$.*

SEPARATING THEOREMS

Fix two theorems P and Q.

How to design the property $\mathcal{P}$ which will separate P from Q?

The LST framework

THE LST FRAMEWORK

Successful approach to separate Ramsey-type statements.

- $\mathsf{EM} \nvdash \mathsf{RT}^2_2$ (Lerman, Solomon & Towsner)
- $\mathsf{DNC} \nvdash \mathsf{RWKL}$ (Flood & Towsner)
- $\mathsf{DNC} \nvdash \mathsf{DNC}_h$ (Flood & Towsner)
- $\mathsf{EM} \nvdash \mathsf{TS}^2$ (P.)
- $\mathsf{TS}^2 \nvdash \mathsf{RT}^2_2$ (P.)
- $\mathsf{RT}^2_2 \nvdash \mathsf{TT}^2_2$ (P.)
- ...

THE LST FRAMEWORK

- Analyse the forcing notion to derive a one-step diagonalization of P
- Generalize the diagonalization to handle multiple iterations of P
- Abstract the diagonalization to have a property independent of the partial order

THE LST FRAMEWORK - ANALYSIS

- Let $\mathbb{P}$ be a forcing notion for constructing solutions to P and G be the generic solution.
- Construct an instance I of Q such that the following set is $\mathbb{P}$-dense for each functional Γ:

$$\{c \in \mathbb{P} : c \text{ forces "}\Gamma^G \text{ is not a solution to } I\text{"}\}$$

THE LST FRAMEWORK - ANALYSIS

By c forces "Γ^G is not a solution to I" we mean

- either c forces Γ^G outputs an invalid sub-solution to I
- or c forces Γ^G is an incomplete solution

How can we ensure this density property?

THE LST FRAMEWORK - ANALYSIS

Given some $c \in \mathbb{P}$ and some Γ, we can usually

- $\emptyset'$-effectively decide whether there is an extension of c such that Γ^G produces more information
- effectively find a finite set of extension candidates if the answer is yes.

THE LST FRAMEWORK - ANALYSIS

The nature of the $\emptyset'$-decidable question strongly depends on the combinatorics of P and Q.

THE LST FRAMEWORK - GENERALIZATION

- The partial order at the next iteration is $\mathbb{P}^{G_0}$, where G_0 is a solution to the first P-instance.
- The same Q-instance I must ensure that following set is $\mathbb{P}^{Y}$-dense for each functional Γ:

$$\{c \in \mathbb{P} : c \text{ forces "} \Gamma^{G_0 \oplus G_1} \text{ is not a solution to } I\text{"}\}$$

THE LST FRAMEWORK - GENERALIZATION

- By extending $c \in \mathbb{P}$, we can obtain more information about $\mathbb{P}^{G_0}$.
- The question over $\mathbb{P}^{G_0}$ is parameterized by G_0.
- We can box the question over $\mathbb{P}^{G_0}$ into a question over $\mathbb{P}$

The questions over $\mathbb{P}$ becomes very complicated

THE LST FRAMEWORK - ABSTRACTION

- The boxing operation shows the ability to answer much more general questions.
- Generic property about all Σ^0_1 formulas.

  "For each Σ^0_1 formula $\varphi(U)$, either $\varphi(I)$ holds or $\varphi(U)$ does not hold for every Q-instance U."
- The property becomes independent of the partial order.

THE LST FRAMEWORK - ABSTRACTION

In particular, for each $c \in \mathbb{P}$,

$$\varphi_{c,\Gamma}(U) = (\exists d \leq c)[d \text{ forces } \Gamma^G \text{ is an invalid solution to } U]$$

For each $c \in \mathbb{P}$, each $p \in \mathbb{P}^{G_0}$ and each Σ^0_1 formula $\varphi(G_0, V)$,

$$\varphi_{c,p}(U) = (\exists d \leq c)(\exists q \leq p)[d \text{ forces } q \notin \mathbb{P}^{G_0} \vee d, q \text{ force } \varphi(G_0, V)]$$

CONCLUSION

- With the LST framework, we have a systematic method to design a property separating two statements.
- The properties are independent of the partial order.
- The resulting properties are genericity notions: not helpful to separate statements from cohesiveness.

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QUESTIONS

Thank you for listening!