THE REVERSE MATHEMATICS OF CAC FOR TREES

JULIEN CERVELLE, WILLIAM GAUDELIER, AND LUDOVIC PATEY

Abstract. CAC for trees is the statement asserting that any infinite subtree of $\mathbb{N}^{<\mathbb{N}}$ has an infinite path or an infinite antichain. In this paper, we study the computational strength of this theorem from a reverse mathematical viewpoint. We prove that CAC for trees is robust, that is, there exist several characterizations, some of which already appear in the literature, namely, the tree antichain theorem (TAC) introduced by Conidis [5], and the statement SHER introduced by Dorais et al. [7]. We show that CAC for trees is computationally very weak, in that it admits probabilistic solutions.

§1. Introduction. In this paper we study the computability-theoretic strength of the statement CAC for trees, which is a variation on the well-studied chain-antichain theorem (CAC). It turns out CAC for trees has different characterizations, making it a robust notion, suitable for future studies in reverse mathematics.

We are going to use two frameworks: reverse mathematics and computable reduction. For a good and more complete introduction to reverse mathematics, see Simpson [17], or Hirschfeldt [11] which also covers the computable reduction and classical results on Ramsey's theorem.

Reverse mathematics is a foundational program which seeks to determine the optimal axioms to prove "ordinary" theorems. It itself uses the framework of subsystems of second-order arithmetic, with a base theory called RCA_0, which informally captures "computable mathematics".

This project started as the study of Ramsey-like theorems for 3-variable forbidden patterns. The attempt to prove Corollary 6.12 naturally led to the study of the SHER principle, already defined by Dorais and al. [7]. Thanks to multiple personal communications with François Dorais, we realized that the SHER principle was closely related to trees, and more precisely, equivalent to the Chain-Antichain principle for trees, a principle studied by Binns et al. in [2]. We later realized that SHER was also equivalent to TAC, an antichain principle for completely branching c.e. trees defined by Conidis [5]. Some of the results are therefore independent rediscoveries of some theorems from [2, 5], but in a more unified setting. The authors are thankful to François Dorais and Chris Conidis for interesting comments and discussions. The authors were partially supported by grant ANR "ACTC" #ANR-19-CE48-0012-01.

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Computable reduction makes precise the idea of being able to computably solve a problem P using another problem Q. A problem is defined as a Π^1_2 formula in the language of second-order arithmetic, thought to be of the form $\forall X(\psi(X) \Longrightarrow \exists Y \varphi(X,Y))$ where ψ and φ are arithmetical formulas. An instance of a problem is a set X verifying $\psi(X)$, and a solution of an instance X is a set Y such that $\varphi(X,Y)$. With this formalism we say that "P is computably reducible to Q" and we write $P \leqslant_c Q$ if for any instance I of P, there is an I-computable instance $\widehat{I}$ of Q, such that, for any solution $\widehat{S}$ of Q, there is an $\widehat{S} \oplus I$-computable solution S of P.

The early study of reverse mathematics has seen the emergence of four subsystems of second-order arithmetic, linearly ordered by the provability relation, such that most of the ordinary theorems are either provable in RCA_0, or equivalent in RCA_0 to one of them. These subsystems, together with RCA_0, form the "Big Five". Among the theorems studied in reverse mathematics, Ramsey's theorem for pairs RT^2_2 plays an important role, since it is historically the first example of a natural statement which does not satisfy this empirical observation. The theorems we study in this paper are all consequences of RT^2_2.

1.1. A chain-antichain theorem for trees. Among the consequences of Ramsey's theorem for pairs, the chain-antichain theorem received a particular focus in reverse mathematics.

Definition 1.1 (CAC, chain-antichain theorem). Every infinite partial order has either an infinite chain or an infinite antichain.

CAC was first studied in [12] by Hirschfeldt and Shore, following a question raised by Cholak, Jockusch and Slaman in [3] (Question 13.8) asking whether or not $\mathsf{CAC} \Longrightarrow \mathsf{RT}^2_2$ over RCA_0, for which they proved the answer is negative (Corollary 3.12). The reciprocal $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \Longrightarrow \mathsf{CAC}$ is easier to obtain, by defining a coloring such that $\{x, y\}$ has color 1 if its elements are comparable, and 0 otherwise. Any homogeneous set for this coloring is either a chain or an antichain, depending on its color.

In this article we focus on the special case where the order is the predecessor relation $\prec$ on a tree.

Definition 1.2 ($\mathsf{CAC\ for\ trees}$). Every infinite subtree of $\mathbb{N}^{<\mathbb{N}}$ has an infinite path or an infinite antichain.

This statement was first introduced by Binns et al. in [2], where the authors showed that every infinite computable tree must have either an infinite computable chain or an infinite Π^0_1 antichain. Furthermore they showed that these bounds are optimal by constructing an infinite computable tree which has no infinite Σ^0_1 chain or antichain. They also showed that $\mathsf{RCA}_0 + \mathsf{WKL} \not\vdash \mathsf{CAC\ for\ binary\ trees}$ (see Definition 2.1).

1.2. Ramsey-like theorems. In [16], Patey identified a formal class of theorems, encompassing several statements surrounding Ramsey's theorem. Indeed, many of them are of the form "for every coloring $f : [\mathbb{N}]^n \to k$ avoiding some set of forbidden patterns, there exists an infinite set $H \subseteq \mathbb{N}$ avoiding some other set of forbidden patterns (relative to f)"

For example the Erdős-Moser theorem (EM) asserts that, "for any coloring $f : [\mathbb{N}]^2 \to 2$, there exists an infinite set $H \subseteq \mathbb{N}$ which is transitive for f", i.e. $\forall i < 2, \forall x<y<z \in H, f(x,y) = i \wedge f(y,z) = i \Longrightarrow f(x,z) = i$. In other terms, we want H to avoid the patterns that would make it not transitive, i.e. $f(x,y) = i \wedge f(y,z) = i \wedge f(x,z) = 1-i$ for any $i < 2$. Another example comes from ADS which is equivalent over RCA_0 to the statement "for any transitive coloring $f : [\mathbb{N}]^2 \to 2$ (i.e. avoiding certain patterns), there exists an infinite set $H \subseteq \mathbb{N}$ which is f-homogeneous" (see [12, Theorem 5.3]). With these definitions one sees that RT^2_2 is equivalent to $\mathsf{EM} + \mathsf{ADS}$ over RCA_0, since EM takes any coloring and "turns it into" a transitive one, and ADS takes any transitive coloring and finds an infinite homogeneous set.

Forbidden patterns on 3 variables and 2 colors are generated by the following three basic patterns:

(1) $f(x,y) = i \wedge f(y,z) = i \wedge f(x,z) = 1-i$
(2) $f(x,y) = i \wedge f(y,z) = 1-i \wedge f(x,z) = i$
(3) $f(x,y) = 1-i \wedge f(y,z) = i \wedge f(x,z) = i$

Avoiding them respectively leads to **transitivity**, **semi-ancestry**, and **semi-heredity** (for the color i). Each of them generates two ramsey-like statements, one restricting the coloring inputted, and one restricting the infinite set outputted, namely "for any coloring avoiding the forbidden pattern, there exists an infinite homogeneous set" and "for any coloring, there exists an infinite set $H \subseteq \mathbb{N}$ which avoids the forbidden pattern". We now survey the known results about these three patterns.

Transitivity. The statement "for any coloring, there exists an infinite set which is transitive for some color" is a weaker version of EM. The Erdös-Moser theorem was proven to be strictly weaker than Ramsey's theorem for pairs over RCA_0 by Lerman, Solomon and Towsner [15, Corollary 1.16]. On the other hand, the statement "for any coloring which is transitive for some color, there exists an infinite homogeneous set" is equivalent to CAC (see [12, Theorem 5.2]), which is also known to be strictly weaker than RT^2_2 over RCA_0 (see Hirschfeldt and Shore [12, Corollary 3.12]).

Semi-ancestry The statement "for any coloring which has semi-ancestry for some color, there exists an infinite homogeneous set" is a consequence of the statement STRIV, defined by Dorais et al. [7, Statement 5.12]), because a 2-coloring is semi-trivial if and only if it has semi-ancestry. And STRIV itself is equivalent to $\forall k, \mathsf{RT}^1_k$ (see the remark below its definition). The statement "for any coloring, there exists an infinite set which has semi-ancestry for some color" is equivalent to RT^2_2 (see Proposition 6.10).

Semi-heredity. The statement "for any coloring which is semi-hereditary for some color, there exists an infinite homogeneous set" is the statement SHER, which was first introduced by Dorais et al. [7, Statement 5.11]. In Section 6, we will show that it is equivalent to $\mathsf{CAC\ for\ trees}$. Finally the statement "for any coloring, there exists an infinite set which is semi-hereditary for some color" is equivalent to RT^2_2 (see Corollary 6.12).

1.3. Notation. The symbols $\exists^\infty x$ and $\forall^\infty x$ are abbreviations for $\forall y, \exists x > y$ and $\exists y, \forall x > y$ respectively, in particular they are the dual of each other.

Given $x, y \in \mathbb{N} \cup \{\pm\infty\}$, we define $[\![x, y]\!] := \{z \in \mathbb{N} \mid z \geqslant x \wedge z \leqslant y\}$, an inequality being strict when its respective bracket is fliped, e.g. $[\![x, y[\![:= \{z \in \mathbb{N} \mid z \geqslant x \wedge z < y\}$. As in set theory, an integer n can also represent the set of all integers strictly smaller to it, i.e. $[\![0, n[\![$. Moreover $\langle -, \dots, - \rangle$ represents a bijection from $\mathbb{N}^n$ to $\mathbb{N}$ (for some n), which verifies $\forall x, y \in \mathbb{N}, \langle x, y \rangle \geqslant \max\{x, y\}$ and which is increasing on each variable.

A **string** is a finite sequence of integers, and the set of all strings is denoted $\mathbb{N}^{<\mathbb{N}}$. In particular, a **binary string** is a finite sequence of 0 and 1, and the set of all binary strings is denoted $2^{<\mathbb{N}}$. The **length** of a given string $\sigma : n \to \mathbb{N}$ is the integer $|\sigma| := n$. The empty string $\varnothing \to \mathbb{N}$ is denoted ε. Given two strings σ and τ of respective length ℓ and m, we define their concatenation as the finite sequence $\sigma \cdot \tau : \ell + m \to \mathbb{N}$ which, given $j < \ell + m$, associates $\sigma(j)$ if $j < \ell$, and $\tau(j - \ell)$ otherwise. So we usually write a string $\sigma : n \to \mathbb{N}$ as $\sigma_0 \cdot \ldots \cdot \sigma_{n-1}$, where $\forall j < n, \sigma_j := \sigma(j)$. We define the partial order on strings "is prefix of", noted $\prec$, by $\sigma \prec \tau \iff |\sigma| < |\tau| \wedge \forall j < |\sigma|, \sigma(j) = \tau(j)$, and denote by $\preccurlyeq$ its reflexive closure.

A tree T is a subset of $\mathbb{N}^{\mathbb{N}}$ which is downward-closed for $\prec$, i.e. $\forall \sigma \in T, \tau \prec \sigma \Longrightarrow \tau \in T$. A binary tree is a subset of $2^{\mathbb{N}}$ which is downward-closed for $\prec$. A subset $S \subseteq T$ of a tree, is a chain when it is linearly ordered for $\prec$, and it is an antichain when its elements are pairwise incomparable for $\prec$.

1.4. Organization of the paper. In Section 2, we prove the robustness of $\mathsf{CAC\ for\ trees}$ in reverse mathematics and over the computable reduction, by proving its equivalence with several variants of the statement. In Section 3, we provide a probabilistic proof of $\mathsf{CAC\ for\ trees}$, and give a precise analysis of this proof in terms of DNC functions. In Section 4 we show that both ADS and EM imply $\mathsf{CAC\ for\ trees}$. In Section 5 we prove there is a computable instance of TAC whose solutions are all of hyperimmune degree, with almost explicit witness. In particular we show a computable instance of TAC can avoid a uniform sequence of Δ^0_2 sets. In Section 6 we show the equivalence between $\mathsf{CAC\ for\ trees}$ and SHER. In Section 7 we explore the relations between stable versions of previously mentioned statements, namely $\mathsf{CAC\ for\ trees}$, ADS and SHER. Leading to the result that $\mathsf{CAC\ for\ stable\ c.e.\ trees}$ admits low solutions. Finally in Section 8 we conclude the paper with remaining open questions and a summary diagram.

§2. Equivalent definitions. In this section we study TAC and variations of $\mathsf{CAC\ for\ trees}$, and prove they are all equivalent. We start with the definitions of these statements.

Definition 2.1 ($\mathsf{CAC\ for\ c.e.\ (binary)\ trees}$). Every infinite c.e. (binary) subtree of $\mathbb{N}^{<\mathbb{N}}$ has an infinite path or an infinite antichain.

Remark 2.2. In the context of reverse mathematics "being c.e. " is a notion that is relative to the model considered, i.e. an object is c.e. when it can be approximated in a c.e. manner by objects from the model. The object

Definition 2.3 (Completely branching tree). A node σ from a tree $T \subseteq 2^{<\mathbb{N}}$ is a **split node** when both $\sigma \cdot 0 \in T$ and $\sigma \cdot 1 \in T$. A tree $T \subseteq 2^{<\mathbb{N}}$ is **completely branching** when, for any of its node σ, if σ is not a leaf then it is a split node.

The following statement was introduced by Conidis [5, Definition 3.2], motivated by the reverse mathematics of commutative noetherian rings.

Definition 2.4 (TAC [5]). Every infinite c.e. subtree of $2^{<\mathbb{N}}$ which is completely branching, contains an infinite antichain.

Conidis proved that TAC follows from CAC over RCA_0, and constructed an instance of TAC, all of whose solutions are of hyperimmune degree. In particular, $\mathsf{RCA}_0 + \mathsf{WKL} \vdash \mathsf{TAC}$. Now we can proceed with the equivalence.

Theorem 2.5. *The following statements are equivalent over* RCA_0 *and computable reduction:*

(1) $\mathsf{CAC\ for\ trees}$
(2) $\mathsf{CAC\ for\ c.e.\ trees}$
(3) $\mathsf{CAC\ for\ c.e.\ binary\ trees}$
(4) TAC

Proof. (2) $\Longrightarrow$ (1) and (2) $\Longrightarrow$ (3) are immediate. (3) $\Longrightarrow$ (4) is Proposition 2.6. (4) $\Longrightarrow$ (2) is Proposition 2.7. (1) $\Longrightarrow$ (2) is Proposition 2.8. ⊣

Proposition 2.6. $\mathsf{RCA}_0 \vdash \mathsf{CAC\ for\ c.e.\ binary\ trees} \Longrightarrow \mathsf{TAC}$ *and* $\mathsf{TAC} \leqslant_c \mathsf{CAC\ for\ c.e.\ binary\ trees}$

Proof. Let $T \subseteq 2^{<\mathbb{N}}$ be an infinite completely branching c.e. tree. By the statement $\mathsf{CAC\ for\ c.e.\ binary\ trees}$, either there is an infinite antichain, or an infinite path P. In the former case, we are done. In the latter case, using the fact that T is completely branching, the set $\{\sigma\cdot(1-i) \mid \sigma\cdot i \prec P\}$ is an infinite antichain of T. ⊣

PROPOSITION 2.7. $\mathsf{RCA}_0 \vdash \mathsf{TAC} \Longrightarrow \mathsf{CAC}$ for c.e. trees *and* CAC for c.e. trees $\leqslant_c \mathsf{TAC}$

PROOF. Let $T \subseteq \mathbb{N}^{\mathbb{N}}$ be an infinite c.e. tree. We can deal with two cases directly: when T has a node with infinitely many immediate children, as they contain a computable infinite antichain; and when T has finitely many split nodes, in which case it has finitely many paths which are all computable. A **split triple** of T as a triple $(\mu, n, m) \in T \times \mathbb{N} \times \mathbb{N}$ such that $\mu, \mu\cdot n, \mu\cdot m \in T$. In particular, μ is a split node in T.

Idea. The general idea is to build greedily a completely branching c.e. tree S by looking for split triples in T, and mapping them to split nodes in S. The correspondance is witnessed by an injective function $f : S \to T$ that will be constructed simultaneously. The main difficulty is that, since T is c.e., a split node ρ can be discovered after μ even though $\rho \prec \mu$, which means that we will not be able to ensure that S is isomorphic to T. In particular, f will not be a tree morphism. However, the only property that needs to be ensured is that for every infinite antichain A of S, the set $f(A)$ will be an infinite antichain of T. To guarantee this, the function f needs to verify

$$(*) \qquad \forall \sigma, \tau \in S, \sigma \perp \tau \Longrightarrow f(\sigma) \perp f(\tau)$$

During the construction, we are going to associate to each node $\sigma \in S$ a set $N_\sigma \subseteq T$, which might decrease in size over time ($N^0_\sigma \supseteq N^1_\sigma \supseteq \dots$), with the property that at every step s, the elements of $\{N^s_\sigma : \sigma \in S\}$ are pairwise disjoint, and their union contains cofinitely many elements of T. The role of N_σ is the following: "if a split triple is found in N_σ, then the nodes in S associated via f must be above σ".

Construction. Initially, $N^0_\varepsilon := T$, $S := \{\varepsilon\}$ and $f(\varepsilon) := \varepsilon$.

At step s, suppose we have defined a finite, completely branching binary tree $S \subseteq 2^{<\mathbb{N}}$, and for every $\sigma \in S$, a set $N^s_\sigma \subseteq T$ such that $\{N^s_\sigma : \sigma \in S\}$ forms a partition of T minus finitely many elements. Moreover, assume we have defined a mapping $f : S \to T$.

Search for a split triple (μ, n_0, n_1) in $\bigcup_{\sigma\in S} N^s_\sigma$, which exists since the c.e. set $\bigcup_{\sigma\in S} N^s_\sigma \setminus T$ is finite. Let $\sigma \in S$ be such that $\mu \in N^s_\sigma$. Let τ be any leaf of S below σ (pick the left-most one, for example). Add $\tau\cdot 0$ and $\tau\cdot 1$ to S, and set $f(\tau\cdot i) = \mu\cdot n_i$ for each $i \in \{0, 1\}$. Note that S is still completely branching.

Then, split N^s_τ into three disjoint subsets N^{s+1}_τ, $N^{s+1}_{\tau\cdot 0}$, $N^{s+1}_{\tau\cdot 1}$ as follows: (for $i < 2$) $N^{s+1}_{\tau\cdot i} := \{\rho \in N^s_\tau \mid \rho \succcurlyeq \mu\cdot n_i\}$ and $N^{s+1}_\tau := \{\rho \in N^s_\tau \mid \rho \perp \mu\}$. Note that these sets do not form a partition of N^s_τ as we missed the nodes in $\{\rho \in N^s_\tau \mid \rho \preccurlyeq \mu\}$, fortunately there are only finitely many of them. Lastly, set $N^{s+1}_\sigma = N^s_\sigma$ for every $\sigma \in S \setminus \{\tau, \tau \cdot 0, \tau \cdot 1\}$.

Verification. First we prove by induction on s that $\forall s \in \mathbb{N}, \forall \sigma \neq \tau \in S, N^s_\sigma \bot N^s_\tau$, i.e. $\forall x \in N^s_\sigma, \forall y \in N^s_\tau, x \bot y$. At step 0, the assertion is trivially verified. At step s, suppose we found the split triple (μ, m_0, m_1) in the set N^s_σ, and that $\forall i < 2, f(\eta \cdot i) = \mu \cdot m_i$ where $\eta \succcurlyeq \sigma$. Since μ was found in N^s_σ, the latter is split into $N^{s+1}_{\eta \cdot 0}$, $N^{s+1}_{\eta \cdot 1}$ and N^{s+1}_σ, the other sets remain identical. By construction, and because they are all subsets of N^s_σ, the assertion holds.

We now prove $(*)$, consider $\sigma, \tau \in S$ such that $\sigma \bot \tau$. WLOG suppose τ was added to S sooner than σ, more precisely $f(\tau)$ appeared (as child in a split triple) at step s in some set N^s_-, so N^{s+1}_τ contains $f(\tau)$ by construction. Since σ is added to S after τ, there exists $\rho \in S$ such that $f(\sigma) \in N^{s+1}_\rho$. By contradiction $f(\sigma) \notin N^{s+1}_\tau$ holds, as otherwise σ would extend τ by construction of S, so $\rho \neq \tau$. Thus by using the previous assertion, we deduce $f(\sigma) \bot f(\tau)$. $\dashv$

Proposition 2.8. $\mathsf{RCA}_0 \vdash$ CAC for trees $\Longrightarrow$ CAC for c.e. trees *and* CAC for c.e. trees $\leqslant_c$ CAC for trees

Proof. Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be a c.e. tree. We define the computable tree $S \subseteq \mathbb{N}^{<\mathbb{N}}$ by $\langle n_0, s_0 \rangle \cdot \ldots \cdot \langle n_{k-1}, s_{k-1} \rangle \in S$ if and only if for all $j < k$, s_j is the smallest integer such that $n_0 \cdot \ldots \cdot n_j \in T[s_j]$, where $T[s_j]$ is the approximation of T at stage s_j.

By CAC for trees on S, there is an infinite chain (resp. antichain). By forgetting the second component of each string, we obtain an infinite chain (resp. antichain) of T. $\dashv$

§3. Probabilistic proofs of CAC for trees. The restriction of CAC to trees yields a strictly weaker statement from the viewpoint the arithmetical bounds in the arithmetic hierarchy. Indeed, by Herrmann [10], there is a computable partial order with no Δ^0_2 infinite chain or antichain, while by Binns et al. in [2], every infinite computable tree must have either an infinite computable chain or an infinite Π^0_1 antichain. In this section, we go one step further in the study of the weakness of CAC for trees by proving it admits probabilistic solutions.

Proposition 3.1. *The measure of the oracles computing a solution for a computable instance of* CAC for trees *is 1.*

Remark 3.2. The above proof is carried purely as a computability statement. In terms of induction we are going to use RT^1_k for an arbitrary k, i.e. we need the strength of $\mathsf{B}\Sigma^0_2$.

Proof. Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be an instance of CAC for trees. We are going to build a decreasing sequence of infinite sets of strings $B_0 \supseteq B_1 \supseteq \ldots$, with

$B_0 = T$, together with an infinite sequence of strings $a_0, a_1, \dots$ such that $A = \{a_i : i \in \mathbb{N}\}$ is an infinite antichain. This construction will work with positive probability, thus we can run infinitely many such constructions in parallel to obtain probability 1.

At step k, assume $B_k \subseteq T$ is defined. Search computably for a finite antichain A_k of size 2^{k+2}. If found, pick an element $a_k \in A_k$ at random. Then define $B_{k+1} := \{y \in B_k \mid a_k \perp y\}$ for the next step.

If the procedure never stops, it yields an infinite antichain $A := \{a_i \mid i \in \mathbb{N}\}$, but two different problems can arise during this construction:

1. Either we cannot find a suitable A_k in B_k
2. Or B_{k+1} is finite, meaning that at some point t we will not be able to find a large enough A_t in it.

If Case 1 happens, then B_k is an infinite computable set in which the sizes of antichains are bounded. In that case, let $C \subseteq B_k$ be a maximal antichain, and consider the infinite set $S := \{\sigma \in B_k \mid \forall \tau \in C, \sigma \npreceq \tau\}$. Then, because C is a maximal antichain, any $\sigma \in S$ is comparable to exactly one element of C. So we can define the coloring $f : S \to C$ that indicates to which element of C a given $\sigma \in S$ is comparable to. Then we can use RT^1_k to obtain and infinite f-homogeneous set $H \subseteq S$ for some color $\tau \in C$. This set is actually an infinite path of T, as otherwise we could find two incomparable elements in it, and ultimately replace τ by them in C, forming a larger antichain.

If Case 2 happens, since B_{k+1} is completely determined by B_k and a_k, it means that we have chosen some "wrong" $a_k \in A_k$. Luckily, there is at most one wrong such element in A_k. Indeed if B_{k+1} is finite, then it means $\forall^\infty y \in B_k, a_k \not\perp y$, more precisely $\forall^\infty y \in B_k, a_k \prec y$. But at most one element of A_k can have this property, as two incomparable elements cannot both verify it. Thus, if we pick at a_k at random over A_k, we have at most $\frac{1}{2^{k+2}}$ chances to make this case happen. Then, the overall probability that this procedure fails is less than $\sum_{k \geqslant 0} \frac{1}{2^{k+2}} = \frac{1}{2}$. Hence we found an antichain with positive probability. ⊣

Very few theorems studied in reverse mathematics admit a probabilistic proof. Proposition 3.1 provides a powerful method for separating the statement $\mathsf{CAC\ for\ trees}$ from many theorems in reverse mathematics. In what follows, AMT stands for the Atomic Model Theorem, studied by Hirschfeldt, Shore and Slaman [13], COH is the cohesiveness principle, defined by Cholak, Jockusch and Slaman [3], and RWKL is the Ramsey-type Weak König's lemma, defined by Flood [9] under the name RKL.

Corollary 3.3. *Over* RCA_0, $\mathsf{CAC\ for\ trees}$ *implies none of* AMT, COH *and* RWKL.

Proof. These three statements have a computable instance such that the measure of the oracles computing a solution is 0, see Astor et al. [1]. ⊣

We are now going to refine Proposition 3.1 by proving that some variant of DNC is sufficient to compute a solution of CAC for trees.

Definition 3.4 (Diagonally non-computable function). A function $f : \mathbb{N} \to \mathbb{N}$ is **diagonally non-computable relative to** X (or $\mathsf{DNC}(X)$) if for every e, $f(e) \neq \Phi_e^X(e)$. Whenever f is dominated by a function $h : \mathbb{N} \to \mathbb{N}$, then we say that f is $\mathsf{DNC}_h(X)$. A Turing degree is $\mathsf{DNC}_h(X)$ if it contains a $\mathsf{DNC}_h(X)$ function.

The following lemma gives a much more convenient way to work with $\mathsf{DNC}_h(X)$ functions.

Lemma 3.5 (Folklore). *Let $A, X \subseteq \mathbb{N}$, the following statements are equivalent:*

(1) *A is of degree $\mathsf{DNC}_h(X)$ for some computable function $h : \mathbb{N} \to \mathbb{N}$.*
(2) *A computes a function $g : \mathbb{N}^2 \to \mathbb{N}$ such that*

$$\forall e, n, |W_e^X| \leqslant n \Longrightarrow g(e,n) \notin W_e^X$$

and which is dominated by a computable function $b : \mathbb{N}^2 \to \mathbb{N}$, i.e.

$$\forall e, n, g(e,n) < b(e,n)$$

Proof. $(2) \Longrightarrow (1)$. Let $i : \mathbb{N} \to \mathbb{N}$ be a computable function such that for any $e \in \mathbb{N}$ and $B \subseteq \mathbb{N}$ we have $\Phi_{i(e)}^B(x)\downarrow \iff x = \Phi_e^B(e)$. Thus

$$W_{i(e)}^B = \begin{cases} \{\Phi_e^B(e)\} & \text{if } e \in B' \\ \varnothing & \text{otherwise} \end{cases}$$

From there, define the A-computable function $f : \mathbb{N} \to \mathbb{N}$ by $f : e \mapsto g(i(e), 1)$. It is $\mathsf{DNC}(X)$ because $g(i(e), 1) \notin W_{i(e)}^X$ since $|W_{i(e)}^X| \leqslant 1$. Moreover, f is dominated by the computable function $e \mapsto h(i(e), 1)$, because h computably dominates g.

$(1) \Longrightarrow (2)$. Let f be a $\mathsf{DNC}_h(X)$ function computed by A. Given the pair e, n, we describe the process that defines $g(e,n)$.

Construction. For each $i < n$ we compute the code $u(e,i)$ of the X-computable function which, on any input, looks for the i^{th} element of W_e^X. If it finds such an element, then it interprets it as an n-tuple $\langle k_0, \dots, k_{n-1}\rangle$ and returns the value k_i. If it never finds such an element, then the function diverges. Finally we define $g : e, n \mapsto \langle f(u(e,0)), \dots, f(u(e,n-1))\rangle$

Verification. First, since f is dominated by h, and since the function $\langle -, \dots, -\rangle$ computing an n-tuple is increasing on each variable, we can dominate g with the computable function

$$b : e, n \mapsto \langle h(u(e,0)), \dots, h(u(e,n-1))\rangle$$

Now, by contradiction, suppose g does not satisfy (2), i.e. suppose there exists e, n such that $|W_e^X| \leqslant n$ but $g(e,n) \in W_e^X$. Because W_e^X has fewer than n elements, we can suppose $g(e,n)$ is the i^{th} one for a some $i < n$. Thus the function $\Phi^X_{u(e,i)}$ is constantly equal to k_i where $g(e,n) = \langle k_0, \dots, k_{n-1}\rangle$, in particular $\Phi^X_{u(e,i)}(u(e,i)) = k_i$. But we also have

$$g(e,n) = \langle f(u(e,0)), \dots, f(u(e,n-1))\rangle$$

implying $f(u(e,i)) = k_i = \Phi^X_{u(e,i)}(u(e,i))$, which is impossible as f is $\mathsf{DNC}_h(X)$. $\dashv$

We are now ready to prove the following proposition. Conidis [5, Theorem 4.5] independently proved the same statement for TAC with a similar construction. Note that by the equivalence of TAC with CAC for trees, Conidis result implies Proposition 3.6.

Proposition 3.6. *Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be an instance of* CAC for trees. *Every set X of degree $\mathsf{DNC}_h(\emptyset')$, with h a computable function, computes a solution of T.*

Proof. First, since X is of degree DNC_h for a computable function h, by Lemma 3.5, it computes a function $g : \mathbb{N}^2 \to \mathbb{N}$ such that $\forall e, n, |W_e^{\emptyset'}| \leqslant n \Longrightarrow g(e,n) \notin W_e^{\emptyset'}$ and which is dominated by a computable function $b : \mathbb{N}^2 \to \mathbb{N}$.

The idea of this proof is the same as in Proposition 3.1, but this time we are going to use g to avoid selecting the potential "bad" element in each finite antichain, i.e. the element which is incompatible with only finitely many strings. For any finite set A, let $\psi_A : \mathbb{N} \to T$ be a bijection such that $\psi_A([\![0, |A|[\![) = A$.

The procedure is the following. Initially, $B_0 := T$. At step k, assume $B_k \subseteq T$ has been defined. To find the desired antichain A_k we use the fix point theorem to find an index e_k such that $\Phi^{\emptyset'}_{e_k}(n)$ is the procedure that halts if it finds an antichain $A \subseteq B_k$ whose size is greater than $b(e_k, 1)$ and $\psi_A(n) \in A$, and finds (using $\emptyset'$) an integer m such that $\forall \ell > m, \psi_A(\ell) \succ \psi_A(n)$.

Define $A_k := A$. By choice of A and e_k,

$$W^{\emptyset'}_{e_k} = \begin{cases} \{\psi_A^{-1}(\sigma_k)\} & \text{if } A_k \text{ has a bad element } \sigma_k \\ \varnothing & \text{otherwise} \end{cases}$$

Note that if we cannot find a big enough antichain, then we can proceed as in the proof of Proposition 3.1 to obtain an infinite path of T, using RT^1_k for a certain k.

Finally we can define $a_k := \psi_A^{-1}(g(e_k, 1))$. Indeed since $|W^{\emptyset'}_{e_k}| \leqslant 1$ by construction, $g(e_k, 1) \notin W^{\emptyset'}_{e_k}$. Moreover $a_k \in A_k$, because $g(e_k, 1) < b(e_k, 1) <$

$|A_k|$. This implies that a_k is not a bad element of A_k, in other words the set $B_{k+1} := \{\tau \in B_k \mid \tau \perp a_k\}$ is infinite. $\dashv$

§4. ADS and EM. Ramsey's theorem for pairs admits a famous decomposition into the Ascending Descending Sequence theorem (ADS) and the Erdős-Moser theorem (EM) over RCA_0. As mentioned in the introduction, both statements are strictly weaker than RT^2_2. Actually, these statements are generally thought of as decomposing Ramsey's theorem for pairs into its disjunctiveness part with ADS, and its compactness part with EM. Indeed, the standard proof of ADS is disjunctive, and does not involve any notion of compactness, while the proof of EM is non-disjunctive and implies RWKL, which is the compactness part of RT^2_2.

ADS and EM are relatively disjoint, in that they are only known to have the hyperimmunity principle as common consequence, which is a particularly weak principle. In this section however, we show that $\mathsf{CAC\ for\ trees}$ follows from both ADS and EM over RCA_0. We shall see in Section 5 that $\mathsf{CAC\ for\ trees}$ implies the hyperimmunity principle.

The following proposition was proved by Dorais (personal communication) for SHER and independently by Conidis [5] for TAC. We adapted the proof of Dorais to obtain the following proposition.

Proposition 4.1. $\mathsf{RCA}_0 \vdash \mathsf{ADS} \implies \mathsf{CAC\ for\ trees}$ *and* $\mathsf{CAC\ for\ trees} \leqslant_c \mathsf{ADS}$

Proof. Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be an infinite tree. Define the total order $<_0$ over T by $\sigma <_0 \tau \iff \sigma \prec \tau \vee (\sigma \perp \tau \wedge \sigma(d) <_{\mathbb{N}} \tau(d))$ where $d := \min\{k \in \mathbb{N} \mid \sigma(k) \neq \tau(k)\}$. By ADS, there is an infinite ascending or descending sequence (σ_i) for $(T, <_0)$.

If it is descending, there are two possibilities. Either $\forall^\infty i, \sigma_i \not\perp \sigma_{i+1}$, which means we eventually have an infinite $\prec$-decreasing sequence of strings, which is impossible. Or $\exists^\infty i, \sigma_i \perp \sigma_{i+1}$, in which case we designate by (h_k) the sequence $(\sigma_{\ell_k+1})_{k\in\mathbb{N}}$ of all such σ_{i+1}, and we show that it is an antichain of T.

To do so, it suffices to prove by induction on m that $\forall m > 0, \forall k, h_k \not\succ h_{k+m}$. When $m = 1$, due to how (σ_i) is structured, we have $\forall k, h_k \succcurlyeq \sigma_{\ell_k} \perp h_{k+1}$. We now consider h_k and $h_{k+(m+1)}$. By induction hypothesis, $h_k \not\succ h_{k+m}$ and $h_{k+m} \not\succ h_{k+(m+1)}$. Moreover since $h_k >_0 h_{k+m} >_0 h_{k+(m+1)}$, there are minima d and e such that $h_k(d) > h_{k+m}(d)$ and $h_{k+m}(e) > h_{k+(m+1)}(e)$. Now $e \leqslant d$ implies $h_{k+(m+1)}(e) < h_{k+m}(e) = h_k(e)$, and $e > d$ implies $h_{k+(m+1)}(d) = h_{k+m}(d) < h_k(d)$; in any case $h_k \not\succ h_{k+(m+1)}$.

Now if the sequence (σ_i) is ascending, we again distinguish two possibilities. Either $\forall^\infty i, \sigma_i \not\perp \sigma_{i+1}$, which means we eventually obtain an infinite path of the tree. Or $\exists^\infty i, \sigma_i \perp \sigma_{i+1}$, in which case we work in the same fashion as in the descending case (designate by (h_k) the sequence $(\sigma_{\ell_k})_{k\in\mathbb{N}}$ of all such σ_i, and show by induction on m that $\forall m > 0, \forall k, h_k \not\prec h_{k+m}$). $\dashv$

PROPOSITION 4.2. $\mathsf{RCA}_0 \vdash \mathsf{EM} \Longrightarrow \mathsf{CAC\ for\ trees}$ *and* $\mathsf{CAC\ for\ trees} \leqslant_c \mathsf{EM}$

PROOF. Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be an infinite tree. We first define a computable bijection $\psi : \mathbb{N} \to T$. To do so, let $\varphi : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ be the bijection $x_0 \cdot \ldots \cdot x_{n-1} \mapsto p_0^{x_0} \times \ldots \times p_{n-1}^{x_{n-1}} - 1$ where p_k is the k^{th} prime number. Use the sequence $(s_n) := (\varphi^{-1}(n))_{n\in\mathbb{N}}$ to define the function $\psi : \mathbb{N} \to T$ which to any number n associates the n^{th} element of the subsequence of (s_n) whose elements are all in T.

Note that, by construction, the range of ψ is infinite and computable. Moreover, if $\sigma \prec \tau$, then $\varphi(\sigma) < \varphi(\tau)$, hence $\psi^{-1}(\sigma) < \psi^{-1}(\tau)$. Also note that the range of ψ is not necessarily a tree.

Let $f : [\mathbb{N}]^2 \to 2$ be the coloring defined by $f(\{x, y\}) = 1$ iff $x <_{\mathbb{N}} y$ and $\psi(x) \prec \psi(y)$ coincide. By EM, there is an infinite transitive set $S \subseteq T$, i.e. $\forall i < 2, \forall x{<}y{<}z \in S, f(x, y) = f(y, z) = i \Longrightarrow f(x, z) = i$

Note that if there are $x < y \in S$ such that $f(x, y) = 0$, then $\forall z > y \in S, f(x, z) = 0$. Indeed given $x < y < z \in S$ such that $f(x, y) = 0$, either $f(y, z) = 0$, and so by transitivity we have $f(x, z) = 0$; otherwise $f(y, z) = 1$, but in that case $f(x, z) \neq 1$ because it is impossible to have $\psi(y) \prec \psi(z)$, $\psi(x) \prec \psi(z)$ and $\psi(x) \perp \psi(y)$.

Now two cases are possible. Either $\exists^\infty j \in \mathbb{N}, f(s_j, s_{j+1}) = 0$, so consider the infinite set A made of all such s_j. Thanks to the previous property, A is f-homogeneous for the color 0, and so $\psi(A)$ is an infinite antichain. Or $\forall^\infty j \in \mathbb{N}, f(s_j, s_{j+1}) = 1$, so there is a large enough $k \in \mathbb{N}$ such that $\psi(s_k) \prec \psi(s_{k+1}) \prec \ldots$, i.e. we found an infinite path. $\dashv$

§5. TAC, lowness and hyperimmunity. Binns et al. in [2] and Conidis [5] independently studied the reverse mathematics of $\mathsf{CAC\ for\ trees}$ and TAC, respectively. Since $\mathsf{CAC\ for\ trees}$ is computably equivalent to TAC and this equivalence also holds in reverse mathematics, each analysis applies to both statements. For example, Binns et al. [2, Theorem 6.4] proved that for any fixed low set L, there is a computable instance of $\mathsf{CAC\ for\ trees}$ with no L-computable solution, while Conidis [5, Corollary 4.16] proved the existence of a computable instance of TAC whose solutions are all of hyperimmune degree. In this section, we prove a general statement regarding TAC (Theorem 5.1) and show that it encompasses both results.

THEOREM 5.1. *Let (A_n) be a uniform sequence of infinite Δ^0_2 sets. There is a computable instance of* TAC *such that no A_n is a solution.*

Proof. First, for any n, let e_n be the index of A_n, i.e. $\Phi_{e_n}^{\emptyset'} = A_n$. We also write $A_n[s] := \Phi_{e_n}^{\emptyset'[s]}[s]$.

Idea. We are going to construct a tree $T \subseteq 2^{<\mathbb{N}}$, such that for each $n \in \mathbb{N}$, there is $\sigma_n \in A_n$ verifying $\sigma_n \notin T$ or $\sigma_n \in T \wedge \forall^\infty \tau \in T, \sigma_n \prec \tau$. These requirements are respectively noted $\mathcal{R}_n$ and $\mathcal{S}_n$, and A_n cannot be an infinite antichain of T if one of them is met.

The sequence (σ_n) is constructed via a movable marker procedure, with steps s and sub-steps $e \in [\![0, s[\![$. At each step s we are going to manipulate an approximation σ_n^s of σ_n, and variables $\widehat{\sigma}_n^s$ that will help us keep track of which requirement is satisfied by σ_n^s.

Construction. At the beginning of each step s, let T_s be the approximation of the tree T defined by $T_s := T_{s-1} \cup \{\tau_s \cdot 0, \tau_s \cdot 1\}$ where τ_s is the leftmost (for example) leaf of T_{s-1} such that $\tau_s \succcurlyeq \widehat{\sigma}_{s-1}^s$. For $s = 0$, we let $T_0 := \{\varepsilon\}$.

At step s, sub-step e, let σ_e^s be the string whose code is the smallest in the uniformly computable set $\{\tau \in A_e[s]{\upharpoonright_s} \mid (\tau \in T_s \wedge \tau \succcurlyeq \widehat{\sigma}_{e-1}^s) \vee \tau \notin T_s\}$ with $\widehat{\sigma}_{-1}^s := \varepsilon$ and σ_e^s is undefined when the set is empty.

Besides, define $\widehat{\sigma}_e^s := \begin{cases} \sigma_e^s & \text{if } \sigma_e^s \in T_s \text{ (and therefore } \sigma_e^s \succcurlyeq \widehat{\sigma}_{e-1}^s) \\ \widehat{\sigma}_{e-1}^s & \text{otherwise} \end{cases}$

Verification. By induction on e, we prove that $\sigma_e := \lim_s \sigma_e^s$ exists and is an element of A_e, also we prove $\widehat{\sigma}_e := \lim_s \widehat{\sigma}_e^s$ exists, and σ_e satisfies $\mathcal{R}_e$ or $\mathcal{S}_e$.

Suppose we reached a step r such that for all $e' < e$ the value of $\sigma_{e'}^r$ and $\widehat{\sigma}_{e'}^r$ have stabilized. And thus, for any step $s > r$, as $\tau_s \succcurlyeq \widehat{\sigma}_{s-1}^s \succcurlyeq \widehat{\sigma}_{e-1}^s = \widehat{\sigma}_{e-1}$, the tree will always be extended with nodes above $\widehat{\sigma}_{e-1}$, implying only a finite part of the tree is not above $\widehat{\sigma}_{e-1}$.

Now suppose k is the smallest code of a string τ such that $(\tau \in T \wedge \tau \succcurlyeq \widehat{\sigma}_{e-1}) \vee \tau \notin T$. Such a string exists because A_e is infinite, whereas the set of strings in T that are below $\widehat{\sigma}_{e-1}$ is not. If $\tau \in T$, then $\exists x, \forall y \geqslant x, \tau \in T_y$, otherwise define $x := 0$. Since A_e is Δ_2^0, there exists $s \geqslant \max\{k+1, r, x\}$ such that $A_e[s]{\upharpoonright_{k+1}}$ has stabilized i.e. $\forall t > s, A_e[t]{\upharpoonright_{k+1}} = A_e[s]{\upharpoonright_{k+1}}$. Thus $\sigma_e^s = \tau$ because $\tau \in A_e{\upharpoonright_{k+1}} = A_e[s]{\upharpoonright_{k+1}} \subseteq A_e[s]{\upharpoonright_s}$. This ensures that for any $t > s$, $\sigma_e^t = \tau$, i.e. $\sigma_e = \tau$.

Finally, we distinguish two cases. Either $\sigma_e \in T$ and so $\exists t, \sigma_e^t \in T_t$, thus $\forall u > t, \widehat{\sigma}_e^u = \sigma_e^u$. So $\mathcal{S}_e$ is satisfied, as cofinitely many nodes of T will be above $\widehat{\sigma}_e = \sigma_e$. Or $\sigma_e \notin T$, in which case, either $\forall t, \sigma_e^t \notin T_t$, implying $\forall t, \widehat{\sigma}_e^t := \widehat{\sigma}_{e-1}^t$ and thus $\mathcal{R}_e$ is satisfied. ⊣

We now show how Theorem 5.1 relates to the result of Binns et al. in [2], that is, the existence, for any fixed low set L, of a computable instance of $\mathsf{CAC\ for\ trees}$ with no L-computable solution.

Lemma 5.2. *For any low set P, the sequence of infinite P-computable sets is uniformly Δ^0_2.*

Proof. Since $P' \leqslant_T \emptyset'$ we can $\emptyset'$-compute the function

$$f(e,x) = \begin{cases} \Phi^P_e(x) & \text{when } \forall y \leqslant x, \Phi^P_e(y)\downarrow \text{ and } \exists y > x, \Phi^P_e(y)\downarrow = 1 \\ 1 & \text{otherwise} \end{cases}$$

Now let $A := \{f(e,x) \mid x \in \mathbb{N}\}$. If Φ^P_e is total and infinite then A is equal to it, so it is P-computable. Otherwise A is cofinite, and in particular it is infinite and P-computable. ⊣

We are now ready to state the result of Binns et al. in [2], but for TAC.

Corollary 5.3. *For any low set P, there exists a computable instance of* TAC *with no P-computable solution.*

Proof. Given P, we can use Lemma 5.2 to obtain a uniform sequence, on which we apply Theorem 5.1. ⊣

The previous corollary is very useful to show that $\mathsf{RCA}_0 + \mathsf{WKL} \nvdash \mathsf{TAC}$, since there exists a model of $\mathsf{RCA}_0 + \mathsf{WKL}$ below a low set. The following corollary will be useful to prove that the result of Binns et al. in [2] implies the result of Conidis.

Corollary 5.4. *There exist a* PA *degree P and an instance of* TAC *with no P-computable solution.*

Proof. It results from the existence of a low PA degree by the low basis theorem, see [14, Corollary 2.2.]. ⊣

The next proposition has two purposes. First, it will be used to show the existence of a computable instance of TAC whose solutions are all of hyperimmune degree (see Theorem 5.6). Second, it shows that, for any such instance, one can choose two specific functionals to witness this hyperimmunity, without loss of generality (see Corollary 5.8).

Proposition 5.5. *Let T be an instance of* TAC. *For any set P of* PA *degree, if T has no P-computable solution, then for any solution $(\sigma_n)_{n\in\mathbb{N}}$, the function $t_{T,(\sigma_n)_{n\in\mathbb{N}}} : n \mapsto \min\{t \mid \sigma_n \in T[t]\}$ or $\ell_{(\sigma_n)_{n\in\mathbb{N}}} : n \mapsto |\sigma_n|$ is hyperimmune.*

Proof. By contraposition, suppose there exists a solution $(\sigma_n)_{n\in\mathbb{N}}$ such that $t_{T,(\sigma_n)_{n\in\mathbb{N}}}$ and $\ell_{(\sigma_n)_{n\in\mathbb{N}}}$ are computably dominated by t and ℓ respectively. Then the set

$$\left\{ (\tau_n)_{n\in\mathbb{N}} \;\middle|\; \begin{array}{l} (\tau_n)_{n\in\mathbb{N}} \text{ is an infinite antichain of } T \\ t_{T,(\tau_n)_{n\in\mathbb{N}}} \leqslant t \text{ and } \ell_{(\tau_n)_{n\in\mathbb{N}}} \leqslant \ell \end{array} \right\}$$

is a Π^0_1 class. It is non-empty because $(\sigma_n)_{n\in\mathbb{N}}$ belongs to it, and to show it is a Π^0_1 class, it can be written as

$$\left\{ (\tau_n)_{n\in\mathbb{N}} \;\middle|\; \forall n,\; \begin{array}{l} \forall m < n, \tau_n \perp \tau_m \\ \tau_n \in T[t(n)] \\ |\tau_n| \leqslant \ell(n) \end{array} \right\}$$

Thanks to ℓ, the number of elements at each level n of the tree associated to this class is computably bounded by $2^{\ell(n)}$, thus it can be coded by a Π^0_1 class of $2^{\mathbb{N}}$. Finally since P is of PA degree, it computes an element of any Π^0_1 class of the Cantor space, hence the result. ⊣

Combining Corollary 5.4 and Proposition 5.5, we obtain the following theorem from Conidis [5].

Theorem 5.6 (Conidis [5]). *There is a computable instance of* TAC *such that each solution is of hyperimmune degree.*

Proof. Let P be of low PA degree. By using Corollary 5.3 we get a computable instance T of TAC with no P-computable solution. Thus, by using Proposition 5.5 we deduce that, for any solution (σ_n), its function $t_{T,(\sigma_n)_{n\in\mathbb{N}}}$ or $\ell_{(\sigma_n)_{n\in\mathbb{N}}}$ is hyperimmune. And $(\sigma_n)_{n\in\mathbb{N}}$ computes both, since T is computable ; meaning it is of hyperimmune degree. ⊣

Corollary 5.7. $\mathsf{RCA}_0 \vdash \mathsf{TAC} \Longrightarrow \mathsf{HYP}$

In his direct proof of Theorem 5.6, Conidis [5] constructed computable instance of TAC and two functionals Φ, Ψ such that for every solution H, either Φ^H or Ψ^H is hyperimmune. Interestingly, Proposition 5.5 can be used to show that Φ and Ψ can be chosen to be $t_{T,-}$ and ℓ_-, without loss of generality.

Corollary 5.8. *For any instance T of* TAC *whose solutions are all of hyperimmune degree, at least one of the function $t_{T,-}$ or ℓ_- is a witness.*

Proof. Let T be an instance of TAC whose solutions are all of hyperimmune degree, and let $(\sigma_n)_{n\in\mathbb{N}}$ be such a solution. By contradiction, if we suppose $t_{T,(\sigma_n)_{n\in\mathbb{N}}}$ and $\ell_{(\sigma_n)_{n\in\mathbb{N}}}$ are both computably dominated, then by Proposition 5.5, T has a P-computable solution. If we choose P to be computably dominated, then it cannot compute a solution of hyperimmune degree, hence a contradiction. ⊣

Note that for every (computable or not) instance of TAC, there is a solution $(\sigma_n)_{n\in\mathbb{N}}$ such that $\ell_{(\sigma_n)_{n\in\mathbb{N}}}$ is dominated by the identify function, by picking any path, and building an antichain along it.

§6. SHER. We have seen in Section 4 that CAC for trees follows from both ADS and EM over RCA_0. The proof of CAC for trees from ADS used only one specific property of the partial order $(T, \prec)$, that we shall refer to as *semi-heredity*. Dorais and al. [7] introduced the principle SHER, which is the restriction of Ramsey's theorem for pairs to semi-hereditary colorings. In this section, we show that the seemingly artificial principle SHER turns out to be equivalent to the rather natural principle $\mathsf{CAC\ for\ trees}$. This equivalence can be seen as more step towards the robustness of $\mathsf{CAC\ for\ trees}$.

Definition 6.1 (Semi-heredity). A coloring $f : [\mathbb{N}]^2 \to 2$ is **semi-hereditary** for the color $i < 2$ if

$$\forall x{<}y{<}z, f(x,z) = f(y,z) = i \Longrightarrow f(x,y) = i$$

The name "semi-heredity" comes from the contraposition of the previous definition $\forall x{<}y{<}z, f(x,y) = 1-i \Longrightarrow f(x,z) = 1-i \vee f(y,z) = 1-i$

Definition 6.2 (SHER, [7]). For any semi-hereditary coloring f, there exists an infinite f-homogeneous set.

The first proposition consists essentially in noticing that, given a set of strings $T \subseteq \mathbb{N}^{<\mathbb{N}}$, the partial order $(T, \prec)$ behaves like a semi-hereditary coloring. The whole technicality of the proposition comes from the definition of an injection $\psi : \mathbb{N} \to T$ with some desired properties.

Proposition 6.3. $\mathsf{RCA}_0 \vdash \mathsf{SHER} \Longrightarrow \mathsf{CAC\ for\ c.e.\ trees}$ *and* $\mathsf{CAC\ for\ c.e.\ trees} \leqslant_c \mathsf{SHER}$

Proof. Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be an infinite c.e. tree. First, let $\varphi : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ the bijection $x_0 \cdot \ldots \cdot x_{n-1} \mapsto p_0^{x_0} \times \ldots \times p_{n-1}^{x_{n-1}} - 1$ where p_k is the k^{th} prime number. Define $\psi : \mathbb{N} \to T$ by letting $\psi(n)$ be the least $\sigma \in T$ (in order of apparition) such that $\phi(\sigma)$ is bigger than $\phi(\psi(0)), \phi(\psi(1)), \ldots, \phi(\psi(n-1))$. Note that, by construction, the range of ψ is infinite and computable. Moreover, if $\sigma \prec \tau$, then $\varphi(\sigma) < \varphi(\tau)$, hence $\psi^{-1}(\sigma) < \psi^{-1}(\tau)$. Also note that the range of ψ is not necessarily a tree.

Let $f : [\mathbb{N}]^2 \to 2$ be the coloring defined by $f(\{x, y\}) = 1$ iff $x <_{\mathbb{N}} y$ and $\psi(x) \prec \psi(y)$ coincide. Let us show that f is semi-hereditary for the color 1. Suppose we have $x < y < z$ and that $f(x, z) = f(y, z) = 1$, i.e. letting $\sigma := \psi(x), \tau := \psi(y), \rho := \psi(z)$ then we have $\sigma \prec \rho$ and $\tau \prec \rho$, thus either $\sigma \prec \tau$ or $\tau \prec \sigma$. But since $x < y$, i.e. $\psi^{-1}(\sigma) < \psi^{-1}(\tau)$, only $\sigma \prec \tau$ can hold due to the above note, meaning $f(x, y) = 1$.

By SHER applied to f, there is an infinite f-homogeneous set H. If it is homogeneous for the color 0, then the set $\psi(H)$ corresponds to an infinite antichain of T. Likewise, if it is homogeneous for the color 1, then the set $\psi(H)$ is an infinite path of T. $\dashv$

We now prove the converse of the previous proposition.

Definition 6.4 (Weak homogeneity). Given a coloring $f : [\mathbb{N}]^2 \to k$, a set $A := \{a_0 < a_1 < \ldots\} \subseteq \mathbb{N}$ is **weakly-homogeneous** for the color $i < k$ if $\forall j, f(a_j, a_{j+1}) = i$

Before proving Proposition 6.6, we need a technical lemma.

LEMMA 6.5 ($\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$). *Let $f : [\mathbb{N}]^2 \to 2$ be a semi-hereditary coloring for the color $i < 2$. For every infinite set $A := \{a_0 < a_1 < \dots\}$ which is weakly-homogeneous for the color i, there is an infinite f-homogeneous subset $B \subseteq A$.*

PROOF (DORAIS). We first show that any a_j falls in one of these two categories:
1. $\forall k > j, f(a_j, a_k) = i$
2. $\exists \ell > j, \begin{cases} \forall k \in]\!]j, \ell[\![, f(a_j, a_k) = i \\ \forall k \geqslant \ell, f(a_j, a_k) = 1 - i \end{cases}$

Indeed, for any $\ell > j$ such that $f(a_j, a_\ell) = i$, by semi-heredity, $f(a_j, a_{\ell-1}) = i$. So with a finite induction we get $\forall k \in]\!]j, \ell]\!], f(a_j, a_k) = i$.

There are now two possibilities. Either there are infinitely many a_j of type 2., in which case one can define an infinite f-homogeneous subset for color $1 - i$ using $\mathsf{B}\Sigma^0_2$. Otherwise, by removing the elements of type 2, the resulting set is f-homogeneous set for the color i. $\dashv$

PROPOSITION 6.6. $\mathsf{RCA}_0 \vdash \mathsf{CAC\ for\ trees} \implies \mathsf{SHER}$ *and* $\mathsf{SHER} \leqslant_c \mathsf{CAC\ for\ trees}$

PROOF. Let $f : [\mathbb{N}]^2 \to 2$ be a semi-hereditary coloring for the color $i < 2$. We begin by constructing a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ defined as $T := \{\sigma_n \mid n \in \mathbb{N}\}$, where σ_n is the unique string which is:
1. strictly increasing, with last element n
2. weak-homogeneous for the color i
3. maximal as a weak-homogeneous set, i.e. $\forall y < \sigma_n(0), f(y, \sigma_n(0)) = 1 - i$ and $\forall j < |\sigma_n| - 1, \forall y \in]\!]\sigma_n(j), \sigma_n(j+1)[\![, f(\sigma_n(j), y) = 1 - i \vee f(y, \sigma_n(j+1)) = 1 - i$

To ensure existence, unicity and that T is a tree, we prove σ_n is obtained via the following effective procedure. Start with the string n. If the string $s_0 \cdot \ldots \cdot s_m$ has been constructed, then look for the biggest integer $j < s_0$ such that $f(j, s_0) = i$. If there is none, the process ends. Else, the process is repeated with the string $j \cdot s_0 \cdot \ldots \cdot s_m$.

The string obtained is maximal by construction. It is unique, because at each step, if there are two (or more) integers $j_0 < j_1$ smaller than s_0 and such that $f(j_0, s_0) = f(j_1, s_0) = i$, then by semi-heredity we have $f(j_0, j_1) = 1$. This means we will eventually add j_0 after j_1. In particular, the string contains all the $j < n$ such that $f(j, n) = i$. Moreover this shows that T is a tree, since the procedure is the same at any point during construction.

Now we can apply $\mathsf{CAC\ for\ trees}$ to T, leading to two possibilities. Either there is an infinite path, which is a weakly-homogeneous set for the color i thanks to condition 2. And so apply Lemma 6.5 (additionally using

Lemma 7.6, as we need $\mathsf{B}\Sigma^0_2$ for the lemma), to obtain a f-homogeneous set for the color i.

Or there is an infinite antichain, which is of the form $(\sigma_{n_j})_{j\in\mathbb{N}}$. Let us show the set $H := \{n_j \mid j \in \mathbb{N}\}$ is f-homogeneous for the color $1-i$. Indeed, if $f(n_s, n_t) = i$ for some $s < t$, then $n_s \in \sigma_{n_t}$, since σ_{n_t} contains all the elements $y < n_t$ such that $f(y, n_t) = i$. But then $\sigma_{n_s} \prec \sigma_{n_t}$, contradicting the fact that $(\sigma_{n_j})_{j\in\mathbb{N}}$ is an antichain. $\dashv$

We end this section by studying RT^2_2 with respect to 3-variables forbidden patterns. As explained in the introduction, there are three basic 3-variables forbidden patterns, yielding the notions of semi-heredity, semi-ancestry and semi-transitivity, respectively. These forbidden patterns induce Ramsey-like statements of the form "for any coloring, there exists an infinite set which avoids some kind of forbidden patterns". This statement applied to semi-transitivity yields a consequence of the Erdos-Moser theorem, known to be strictly weaker than Ramsey's theorem for pairs over RCA_0. We now show that the two remaining forbidden patterns yield statements equivalent to RT^2_2. This completes the picture of the reverse mathematics of Ramsey-like theorems for 3-variable forbidden patterns.

Before proving RT^2_2 from the Ramsey-like statement about semi-ancestry over RCA_0, we need to prove that this statement implies $\mathsf{B}\Sigma^0_2$. This is done by proving the following principle.

Definition 6.7 (D^2_2). Every Δ^0_2 set admits an infinite subset in it or its complement.

Proposition 6.8. *The statement "for any coloring, there exists an infinite set which has semi-ancestry for some color" implies D^2_2 over RCA_0.*

Proof. Let A be a Δ^0_2 set whose approximations are $(A_t)_{t\in\mathbb{N}}$. We define the coloring $f(x, y) := x \in A_y$, and use the statement of the proposition to obtain an infinite set B that has semi-ancestry for some color.

If B has semi-ancestry for the color 1, then $\forall x{<}y{<}z \in B, x \in A_y \wedge x \in A_z \Longrightarrow y \in A_z$. Now either $B \subseteq \overline{A}$ and we are done. Or $\exists x \in A \cap B$, which means $\forall^\infty y \in B, x \in A_y$, implying that $\forall^\infty y > x \in B, \forall z > y \in B, y \in A_z$ by semi-ancestry, i.e. $\forall^\infty y > x \in B, y \in A$. So we can compute a subset H of B which is infinite and such that $H \subseteq A$.

This argument also works when B has semi-ancestry for the color 0, we just need to switch A and $\overline{A}$, as well as $\in$ and $\notin$, when needed. $\dashv$

Corollary 6.9. *The statement "for any coloring, there exists an infinite set which has semi-ancestry for some color" implies $\mathsf{B}\Sigma^0_2$ over RCA_0.*

Proof. Immediate, since $\mathsf{RCA}_0 \vdash \mathsf{D}^2_2 \Longrightarrow \mathsf{B}\Sigma^0_2$, see [4, Theorem 1.4]. $\dashv$

PROPOSITION 6.10. *The statement "for any coloring, there exists an infinite set which has semi-ancestry for some color" implies* RT^2_2 *over* RCA_0 *and over the computable reduction.*

PROOF. Let $f : [\mathbb{N}]^2 \to 2$ be a coloring. We can apply the statement to obtain an infinite set A which has semi-ancestry for the color i, i.e. $\forall x{<}y{<}z \in A, f(x,y) = i \wedge f(x,z) = i \Longrightarrow f(y,z) = i$. There are two possibilities. Either there exists $a \in A$ such that $\exists^\infty b > a \in A, f(a,b) = i$, in which case all such elements b form an infinite f-homogeneous set due to the property of A. Otherwise any $a \in A$ verifies $\forall^\infty b > a, f(a,b) = 1-i$, i.e. all the elements of A have a limit color equal to $1-i$ for the coloring $f\restriction_{[A]^2}$. Thus we can use $\mathsf{B}\Sigma^0_2$ (Corollary 6.9) to compute an infinite homogeneous set. ⊣

The proof that the Ramsey-like statement about semi-heredity implies Ramsey's theorem for pairs is indirect, and uses the Ascending Descending Sequence principle.

PROPOSITION 6.11. *The statement "for any coloring, there exists an infinite set which is semi-hereditary for some color" implies* ADS *over* RCA_0 *and over the computable reduction.*

PROOF. Let $\mathcal{L} = (\mathbb{N}, <_{\mathcal{L}})$ be a linear order. Let $f : [\mathbb{N}]^2 \to 2$ be the coloring defined by $f(\{x,y\}) = 1$ iff $<_{\mathcal{L}}$ and $<_{\mathbb{N}}$ coincide on $\{x,y\}$. By the statement of the proposition, there is an infinite set H on which the coloring is semi-hereditary for some color i.

Before continuing, note that if there is a pair $x < y \in H$ such that $f(x,y) = 1-i$ then $\forall z > y \in H, f(x,z) = 1-i$. Indeed, either $f(y,z) = 1-i$, in which case by transivity of $<_{\mathcal{L}}$ we have $f(x,z) = 1-i$. Or $f(y,z) = i$, in which case by semi-heredity $f(x,z) = 1-i$, because otherwise it would imply that $f(x,y) = i$.

Now if $\exists^\infty x \in H, \exists y > x \in H, f(x,y) = 1-i$, then we construct an infinite decreasing sequence by induction. Suppose the sequence $x_0 >_{\mathcal{L}} \ldots >_{\mathcal{L}} x_{n-1}$ has already be constructed, and we have $m \in H$ such that $f(x_{n-1}, m) = 1-i$, then we look for a $\{x < y\}$ such that $x > m$ and $f(x,y) = 1-i$. By the previous remark we have that $f(x_{n-1}, x) = 1-i$, i.e. $x_{n+1} >_{\mathcal{L}} x$, so we extend the sequence with x and redefine $m := y$.

Else $\forall^\infty x \in H, \forall y > x \in H, f(x,y) = i$, and so getting rid of finitely many elements yields an infinite increasing sequence. ⊣

COROLLARY 6.12. *The statement "for any coloring, there exists an infinite set which is semi-hereditary for some color" implies* RT^2_2 *over* RCA_0.

PROOF. This comes from the fact that $\mathsf{RT}^2_2 \iff S + \mathsf{SHER}$ by definition (with S denoting the statement in question). And we have $\mathsf{RCA}_0 \vdash S \Longrightarrow \mathsf{SHER}$ by using Proposition 6.11, Proposition 4.1 and Proposition 6.6. ⊣

Remark 6.13. Let S denote the statement "for any coloring, there exists an infinite set which is semi-hereditary for some color". The proof that S implies RT^2_2 over RCA_0 involves two applications of S. The first one to obtain an infinite set over which the coloring is semi-hereditary, and a second one to solve SHER using the fact that S implies ADS, which itself implies SHER. It is unknown whether RT^2_2 is computably reducible to S.

§7. Stable counterparts: SADS and CAC for stable c.e. trees. Cholak, Jockusch and Slaman [3] made significant progress in the understanding of Ramsey's theorem for pairs by dividing the statement into a stable and a cohesive part.

Definition 7.1. A coloring $f : [\mathbb{N}]^2 \to k$ is **stable** if for every $x \in \mathbb{N}$, $\lim_y f(x, y)$ exists. A linear order $\mathcal{L} = (\mathbb{N}, <_{\mathcal{L}})$ is **stable** if it is of order type $\omega + \omega^*$.

We call SRT^2_k and SADS the restriction of RT^2_k and ADS to stable colorings and stable linear orders, respectively. Given a linear order $\mathcal{L} = (\mathbb{N}, <_{\mathcal{L}}$), the coloring corresponding to the order is stable iff the linear order is of order type $\omega + \omega^*$, or $\omega + k$ or $k + \omega^*$. In particular, SRT^2_2 implies SADS over RCA_0.

In this section, we study the stable counterparts of CAC for trees and SHER, and prove that they are equivalent over RCA_0. We show SADS implies CAC for stable c.e. trees over RCA_0. It follows in particular that every computable instance of CAC for stable c.e. trees admits a low solution.

Definition 7.2 (Stable tree, Dorais [6]). A tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is **stable** when for every $\sigma \in T$ either $\forall^\infty \tau \in T, \sigma \perp \tau$ or $\forall^\infty \tau \in T, \sigma \not\perp \tau$

Note that any stable finitely branching tree admits a unique path.

Proposition 7.3. $\mathsf{RCA}_0 \vdash \mathsf{SADS} \Longrightarrow \mathsf{CAC}$ for stable c.e. trees

Remark 7.4. In the proof below, we use the fact that $\mathsf{RCA}_0 \vdash \mathsf{SADS} \Longrightarrow \mathsf{B}\Sigma^0_2$. This is because $\mathsf{RCA}_0 \vdash \mathsf{SADS} \Longrightarrow \mathsf{PART}$ ([12, Proposition 4.6]) and $\mathsf{RCA}_0 \vdash \mathsf{PART} \iff \mathsf{B}\Sigma^0_2$ ([4, Theorem 1.2]).

Proof. Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be an infinite c.e. tree which is stable.

Consider the total order $<_0$, defined by $\sigma <_0 \tau \iff \sigma \prec \tau \vee (\sigma \perp \tau \wedge \sigma(d) < \tau(d))$ where $d := \min\{y \mid \sigma(y) \neq \tau(y)\}$. We show that it is of type $\omega + \omega^*$, i.e.

$$\forall \sigma \in T, (\forall^\infty \tau \in T \sigma <_0 \tau) \ \vee \ (\forall^\infty \tau \in T, \tau <_0 \sigma)$$

So let $\sigma \in T$, there are two possibilities. Either $\forall^\infty \tau \in T, \sigma \not\perp \tau$, meaning we even have $\forall^\infty \tau \in T, \sigma \prec \tau$, which directly implies $\forall^\infty \tau \in T \sigma <_0 \tau$.

Or $\forall^\infty \tau \in T, \sigma \perp \tau$. In which case consider all the nodes τ successors of a prefix of σ but not prefix of σ, there are finitely many of them, because there are finitely many prefixes of σ and no infinitely-branching node (WLOG, as otherwise there would be a computable infinite antichain). So we can apply the pigeon-hole principle, by using $\mathsf{B}\Sigma^0_2$, to deduce that there is a certain τ which has infinitely many successors.

Moreover, by stability of T there cannot be another such node. Indeed, by contradiction, if there were two such nodes τ and τ', then we would have that $\exists^\infty \eta \in T, \eta \perp \tau$, because the successors of τ' are incomparable to τ. And since τ already verifies $\exists^\infty \eta \in T, \eta \succ \tau$, contradicting the stability of T.

Therefore we have that $\forall^\infty \eta \in T, \eta \succ \tau$, and so depending on whether $\tau <_0 \sigma$ or $\sigma <_0 \tau$, we obtain that $\forall^\infty \eta \in T, \eta <_0 \sigma$ or $\forall \eta \in T, \sigma <_0 \eta$ respectively. From there we can apply SADS and the proof is exactly like in Proposition 4.1. $\dashv$

Corollary 7.5. CAC for stable c.e. trees *admits low solutions.*

Proof. This comes from the fact that any instance of SADS has a low solution, as proven in [12, Theorem 2.11]. $\dashv$

The proof that SHER follows from CAC for stable trees will use $\mathsf{B}\Sigma^0_2$. We therefore first prove that CAC for stable trees implies $\mathsf{B}\Sigma^0_2$ over RCA_0.

Lemma 7.6. $\mathsf{RCA}_0 \vdash \mathsf{CAC}$ for stable trees $\Longrightarrow \forall k, \mathsf{RT}^1_k$

Proof. Let $f : \mathbb{N} \to k$ be a coloring, there are two possibilities. Either $\exists i < k, \exists^\infty x, f(x) = i$, in which case there is an infinite computable f-homogeneous set. Otherwise $\forall i < k, \forall^\infty x, f(x) \neq i$, in which case we consider the infinite tree

$$T := \{\sigma \in \mathsf{Inc} \mid \text{letting } x := \max \sigma, \forall y \leqslant x, f(y) = f(x) \Longrightarrow y \in \mathsf{ran}\sigma\}$$

where Inc is the set of strictly increasing strings of $\mathbb{N}^{<\mathbb{N}}$.

By hypothesis, for every node $\sigma \in T$, $\forall^\infty \tau \in T, \sigma \perp \tau$, otherwise T would have an infinite T-computable path. Thus, T is stable. Moreover, every antichain is of size at most k, thus T is a stable tree with no infinite path and no infinite antichain, contradicting CAC for stable trees. $\dashv$

Proposition 7.7. $\mathsf{RCA}_0 \vdash \mathsf{CAC}$ for stable trees $\Longrightarrow \mathsf{SHER}$ for stable colorings

Proof. Let $f : [\mathbb{N}]^2 \to 2$ be a stable coloring, semi-hereditary for the color i. We distinguish two cases. Either there are finitely many integers with limit color i, meaning we can ignore them and use $\mathsf{B}\Sigma^0_2$ (Lemma 7.6) to compute an infinite homogeneous set (see [8, Proposition 6.2]). Otherwise there are infinitely many integers whose limit color is i, in which case we use the same proof as in Proposition 6.6, but we must prove the tree T we construct is stable. So let σ_n be a node of this tree.

Suppose first n has limit color i. Let p be sufficiently large so that $f(n,p) = i$. As explained in Proposition 6.6, σ_p contains all the integers $m < p$ such that $f(m,p) = i$. It follows that $n \in \sigma_p$. Moreover, if $n \in \sigma_p$, then $\sigma_n \preceq \sigma_p$. Thus, $\forall^\infty p, \sigma_n \prec \sigma_p$.

Suppose now n has limit color $1-i$, then since there are infinitely many integers with limit color i, there is one such that $p > n$. In particular, σ_p verifies $\forall^\infty \tau \in T, \sigma_p \prec \tau$. Thus, if $\sigma_n \prec \sigma_p$ then $\forall^\infty \tau \in T, \sigma_n \prec \tau$, and if $\sigma_n \perp \sigma_p$ then $\forall^\infty \tau \in T, \sigma_n \perp \tau$. $\dashv$

Proposition 7.8. $\mathsf{RCA}_0 \vdash \mathsf{SHER\ for\ stable\ colorings} \Longrightarrow \mathsf{CAC\ for\ stable\ c.e.\ trees}$

Proof. Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be an infinite stable c.e. tree. The proof is the same as in Proposition 6.3, but we must verify that the coloring $f : [\mathbb{N}]^2 \to 2$ defined is stable. Given $x \in \mathbb{N}$, we claim that $\exists i < 2, \forall^\infty y f(x,y) = i$. Since T is stable, either $\forall^\infty y, \psi(x) \not\perp \psi(y)$ or $\forall^\infty y, \psi(x) \perp \psi(y)$ holds. In the first case $\forall^\infty y, f(x,y) = 1$, and in the second one $\forall^\infty y, f(x,y) = 0$. Thus the coloring is stable, and the proof can be carried on. $\dashv$

Corollary 7.9. *The following are equivalent over* RCA_0*:*

(1) CAC for stable trees
(2) CAC for stable c.e. trees
(3) SHER for stable colorings

§8. Conclusion.

The following figure summarizes the implications proved in this paper. All implications hold both in RCA_0 and over the computable reduction.

$$
\begin{tikzcd}
\mathsf{EM} \arrow[r] & \mathsf{CAC\ for\ trees} \arrow[d, bend right] \arrow[dr] & \mathsf{ADS} \arrow[l] \\
\mathsf{TAC} \arrow[r] \arrow[d] & \mathsf{CAC\ for\ c.e.\ trees} \arrow[u, bend right] \arrow[d] & \mathsf{SHER} \arrow[l] \\
\mathsf{HYP} & \mathsf{CAC\ for\ c.e.\ binary\ trees} \arrow[ul] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathsf{CAC\ for\ stable\ c.e.\ trees} \arrow[d] & \mathsf{SADS} \arrow[l] \\
\mathsf{CAC\ for\ stable\ trees} \arrow[r] & \mathsf{SHER\ for\ stable\ colorings} \arrow[ul]
\end{tikzcd}
$$

Recall that, by Corollary 5.3, for every fixed low set X, there is a computable instance of TAC with now X-computable solution. By computable equivalence, this property also holds for CAC for trees. It is however unknown whether Corollary 5.3 can be improved to defeat all low sets simultaneously.

Question 8.1. Does every computable instance of CAC for trees admits a low solution?

Note that by Corollary 7.5, any negative witness to the previous question would yield a non-stable tree.

We have also seen by Proposition 3.6 that for every computable instance T of CAC for trees, every computably bounded DNC function relative to $\emptyset'$ computes a solution to T. The natural question would be whether we can improve this result to any DNC function relative to $\emptyset'$.

Question 8.2. Is there some X such that for every computable instance T of CAC for trees, every DNC function relative to X computes a solution to T?

Note that in the case of a computable set X, the answer is negative, as there exist DNC functions of low degree.

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UNIV PARIS EST CRETEIL, LACL, F-94010 CRETEIL, FRANCE
E-mail: julien.cervelle@u-pec.fr
URL: https://jc.lacl.fr

UNIV PARIS EST CRETEIL, LACL, F-94010 CRETEIL, FRANCE
E-mail: william.gaudelier@gmail.com

CNRS, ÉQUIPE DE LOGIQUE
UNIVERSITÉ DE PARIS
PARIS, FRANCE
E-mail: ludovic.patey@computability.fr
URL: http:/ludovicpatey.com