# $\{\Pi^0_1$ CLASSES AND PA DEGREES$\}$

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## TREES

### Tree

A *binary tree* is a set $T \subseteq 2^{<\mathbb{N}}$ closed under prefix, i.e., for all $\sigma \in T$, and $\tau \preceq \sigma$, $\tau \in T$.

### Path

A *path* through a binary tree $T \subseteq 2^{<\mathbb{N}}$ is a sequence $P \in 2^{\mathbb{N}}$ such that for all $n \in \mathbb{N}$, $P(n) \in T$.
We denote by $[T]$ the class of paths of $T$.

### Weak König's lemma

Let $T \subseteq 2^{<\mathbb{N}}$ be an infinite binary tree. Then, $[T]$ is non-empty.

## EXTENDIBLE NODES

### Extendible node

A node $\sigma$ of a binary tree $T$ is *extendible* in $T$ if the set $\{\tau \in T : \sigma \preceq \tau\}$ is infinite.

### Computable vs $\Pi^0_1$ binary tree

Let $T$ be a $\Pi^0_1$ binary tree. There exists a computable binary tree $S$ such that $[T] = [S]$.

The set of extendible nodes of a computable binary tree $T \subseteq 2^{<\mathbb{N}}$ is a $\Pi^0_1$ binary tree $S \subseteq T$ such that $[S] = [T]$.

## PERFECT TREES

### Perfect trees

A non empty binary tree $T$ is *perfect* if any node in $T$ has two incompatible extensions in $T$, i.e., extensions incomparable w.r.t. the partial order $\prec$.

### Isolated points

Let $\mathcal{A} \subseteq 2^{\mathbb{N}}$. An element $X \in \mathcal{A}$ is *isolated* in $\mathcal{A}$ if there is a $\sigma \prec X$ such that $[\sigma] \cap \mathcal{A} = \{X\}$.

### Perfect classes

A non-empty class $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is perfect if it is closed and has no isolated point, or equivalently, $\mathcal{F} = [T]$ for a perfect tree $T$.

## TOPOLOGY ON CANTOR SPACE

### Notations

Given a string $\sigma \in 2^{<\mathbb{N}}$ we write $[\sigma]$ for the class $\{X \in 2^{\mathbb{N}} : \sigma \preceq X\}$. We call *cylinder* a class of the form $[\sigma]$.

Given a set $W \subseteq 2^{<\mathbb{N}}$ we will write $[W]$ to denote its corresponding open class i.e., the class $\bigcup_{\sigma \in W} [\sigma]$.

### Open classes

The *open* classes of Cantor space are arbitrary unions of cylinders, i.e., sets of the form $\bigcup_{\sigma \in W} [\sigma]$ for a set $W \subseteq 2^{<\mathbb{N}}$.

### Closed classes

A class $\mathcal{P} \subseteq 2^{\mathbb{N}}$ is *closed* iff it is the complement of an open class, or equivalently iff $\mathcal{P} = [T]$ for a binary tree $T$.

### Compactness

In the Cantor Space, the compact classes are exactly the closed classes. In particular, the intersection of any decreasing sequence of non-empty closed classes is non-empty.

### Continuous functions

A (possibly partial) function $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is *continuous* in $X \in \operatorname{dom} f$ if for any cylinder $[\tau]$ containing $f(X)$, there exists a cylinder $[\sigma]$ containing $X$ such that $f([\sigma]) \subseteq [\tau]$.

## $\Sigma^0_1$ CLASSES

### Definition

A class $\mathcal{U} \subseteq 2^{\mathbb{N}}$ is $\Sigma^0_1$ if there exists a $\Sigma^0_1$ set $W \subseteq 2^{<\mathbb{N}}$ such that $\mathcal{U} = [W]$.

### Characterization

A class $\mathcal{P} \subseteq 2^{\mathbb{N}}$ is $\Sigma^0_1$ iff $\mathcal{P}$ can be written of the form $\{X \in 2^{\mathbb{N}} : \exists n R(X \upharpoonright_n)\}$ for a computable predicate $R \subseteq 2^{<\mathbb{N}}$.

### Closure properties

The $\Sigma^0_1$ are closed under finite intersection, and uniform countable unions.

## $\Pi^0_1$ CLASSES

### Definition

A class $\mathcal{P} \subseteq 2^{\mathbb{N}}$ is $\Pi^0_1$ if its complement is $\Sigma^0_1$.

### Characterizations

A class $\mathcal{P} \subseteq 2^{\mathbb{N}}$ is $\Pi^0_1$ iff $\mathcal{P} = [T]$ for a computable binary tree $T \subseteq 2^{<\mathbb{N}}$, or equivalently iff $\mathcal{P} = \{X \in 2^{\mathbb{N}} : \forall n R(X \upharpoonright_n)\}$ for a computable predicate $R \subseteq 2^{<\mathbb{N}}$.

### Closure properties

The $\Pi^0_1$ are closed under finite unions, and uniform countable intersections.

### Computable members of $\Pi^0_1$ classes

There are non-empty $\Pi^0_1$ classes that do not contain any computable set.

Let $\mathcal{P} = \{X\}$ be a $\Pi^0_1$ class. Then, $X$ is computable and is called a $\Pi^0_1$ *singleton*.

The isolated points of any $\Pi^0_1$ class are computable.

Any non-empty countable $\Pi^0_1$ class has a computable member.

## BASIS THEOREMS

### Low basis theorem

Every non-empty $\Pi^0_1$ class contains a member of low degree.

### Computably dominated basis theorem

Every non-empty $\Pi^0_1$ class contains a member of computably dominated degree.

### Cone avoidance basis theorem

Suppose $C \not\leq_T \emptyset$. Every non-empty $\Pi^0_1$ class contains a member $X$ such that $C \not\leq_T X$.

## PA DEGREES

### PA degree

A set is of *PA degree* if and only if it computes a set in each non-empty $\Pi^0_1$ class.

### $\mathrm{DNC}_2$ characterization

A set $X$ is of PA degree iff it computes a function $f : \mathbb{N} \to 2$ such that $f(n) \neq \phi_n(n)$ for all $n$.

### Degree spectrum

The degree spectrum of a class $\mathcal{P} \subseteq 2^{\mathbb{N}}$ is the class $\deg \mathcal{P} = \{\deg_T X : X \in [P]\}$.

There is a $\Pi^0_1$ class whose degree spectrum correspond to the PA degrees.

### PA or high

A set is of PA or high degree iff it computes a sequence containing all the computable sets.

## FINITELY-BRANCHING TREES

### Tree

A *tree* is a set $T \subseteq \mathbb{N}^{<\mathbb{N}}$ closed under prefix, i.e., for all $\sigma \in T$, and $\tau \preceq \sigma$, $\tau \in T$.

A tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is *finitely-branching* if every $\sigma \in T$ has finitely many immediate successors in $T$.

### König's lemma

Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be an infinite, finitely-branching tree. Then, $[T]$ is non-empty.

### $\Delta^0_2$ binary trees

The degree spectra of finitely-branching computable trees and those of $\Delta^0_2$ binary trees coincide.

### Computably bounded trees

A tree $T$ is *computably bounded* if there exists a computable function $g : \mathbb{N} \to \mathbb{N}$ such that for all $\sigma \in T$ and $n < |\sigma|$, $\sigma(n) < g(n)$.

The degree spectra of computably bounded computable trees and those of computable binary trees coincide.