{SECOND-ORDER ARITHMETIC}

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FORMULAS

Language

The *language* $\mathcal{L}_{Z_2}$ *of second order arithmetic* consists of :

- first order variable symbols: x, y, z, ...
- second order variable symbols: X, Y, Z, ...
- logicals symbols: $\wedge, \vee, \rightarrow, \neg, \exists, \forall$
- parenthesis: ( )
- two binary function symbols on the integers: $+$ and $\times$
- two binary relation symbols on the integers: $=$ and $<$
- the membership relation symbol: $\in$
- two constant symbols: $\dot{0}$ and $\dot{1}$

First-order terms

The *first-order terms* of Z_2 are defined inductively as follows :

- A first-order variable or constant is a first-order term.
- If t_1 and t_2 are first-order terms, then so are $(t_1 + t_2)$ and $(t_1 \times t_2)$

Second-order terms

The *second-order terms* of Z_2 are simply the second-order variables.

Atomic formulas

The *atomic formulas* of second-order arithmetic consists of all the $t_1 = t_2$, $t_1 < t_2$ and $t_1 \in X$ for all t_1, t_2 first-order terms and X second order term.

Formulas

The *formulas* of second-order arithmetic are defined inductively as follows :

- Every atomic formula is a formula.
- If F_1 and F_2 are formulas, then so are $(F_1 \wedge F_2)$, $(F_1 \vee F_2)$, $(F_1 \rightarrow F_2)$ and $\neg F_1$.
- If F is a formula, x a first-order symbol and X a second-order symbol then $\forall x F$, $\exists x F$, $\forall X F$ and $\exists X F$ are formulas.

Z_2 THEORY

Robinson arithmetic

Robinson's theory of arithmetic denoted by Q consists of the following eight axioms :

1. $\forall x\ \neg(x + \dot{1} = \dot{0})$
2. $\forall x(x = \dot{0} \vee \exists y\ (x = y + \dot{1}))$
3. $\forall x \forall y\ (x + \dot{1} = y + \dot{1} \rightarrow x = y)$
4. $\forall x\ (x + \dot{0} = x)$
5. $\forall x \forall y\ (x + (y + \dot{1}) = (x + y) + \dot{1})$
6. $\forall x\ (x \times \dot{0} = \dot{0})$
7. $\forall x \forall y\ (x \times (y + 1) = (x \times y) + x)$
8. $\forall x \forall y\ (x < y \leftrightarrow (\exists z\ (z \neq \dot{0} \wedge x + z = y)))$

Comprehension scheme

The *comprehension scheme* consists of all the formulas of the form:

$$\exists X \forall y\ (y \in X \leftrightarrow F(y))$$

for any formula $F(x)$ with X not free in F.

Induction axiom

The *induction axiom* is the following :

$$\forall X\ ([\dot{0} \in X \wedge \forall y\ (y \in X \rightarrow y + \dot{1} \in X)] \rightarrow \forall z\ z \in X)$$

Combined with the comprehension scheme, the induction axiom gives us the induction scheme :

$$(F(\dot{0}) \wedge (\forall x\ (F(x) \rightarrow F(x + \dot{1})))) \rightarrow \forall x\ F(x)$$

for any formula $F(x)$.

Z_2 theory

We note by Z_2 the theory of second-order arithmetic composed of Q, the comprehension scheme and the induction axiom.

SEMANTICS

Henkin structure

A *(Henkin) structure* in $\mathcal{L}_{Z_2}$ is given by a tuple $\mathcal{M} = (M, S, +^{\mathcal{M}}, \times^{\mathcal{M}}, <^{\mathcal{M}}, 0^{\mathcal{M}}, 1^{\mathcal{M}})$ where :

- M and S are disjoints sets with $S \subseteq \mathcal{P}(M)$.
- $+^{\mathcal{M}}, \times^{\mathcal{M}}$ are functions from $M \times M$ to M.
- $<^{\mathcal{M}}$ is a binary relation on M.
- $0^{\mathcal{M}}$ and $1^{\mathcal{M}}$ are two elements from M.
- $=$ is the equality on M

A *model* of Z_2 is an Henkin structure in which the axioms of Z_2 are verified.

(Non)-standard integers

Let $\mathcal{M} = (M, S, +^{\mathcal{M}}, \times^{\mathcal{M}}, <^{\mathcal{M}}, 0^{\mathcal{M}}, 1^{\mathcal{M}})$ be a model of Z_2.

The elements of $\omega = \{\dot{0}^{\mathcal{M}}, \dot{1}^{\mathcal{M}}, \dot{1}^{\mathcal{M}} + \dot{1}^{\mathcal{M}}, \dots\} \subseteq S$ are called the *standard integers*. The elements not in ω are the *non-standard integers*.

A model with non-standard integers is itself called *non-standard*.

Full structure

A Henkin structure $\mathcal{M} = (M, S, +^{\mathcal{M}}, \times^{\mathcal{M}}, <^{\mathcal{M}}, 0^{\mathcal{M}}, 1^{\mathcal{M}})$ is said to be *full* if $S = \mathcal{P}(M)$.

A full model of Z_2 contains no non-standard integers and is isomorphic to the standard integers.

There is no completeness theorem for full structures.

ω-structure

An *ω-structure* is a structure $\mathcal{M} = (\omega, S, +, \times, <, 0, 1)$ where ω is the set of standard integers, $S \subseteq \mathcal{P}(\omega)$, $+$ and $\times$ the addition on multiplication on ω and $<$ the natural order.

MAIN SUBSYSTEMS

RCA_0

We denote by RCA_0 the theory composed of Robinson arithmetic Q, the induction scheme for Σ^0_1 formulas, and the following Δ^0_1*-comprehension scheme*:

$$\forall y(F(y) \leftrightarrow G(y)) \rightarrow \exists X \forall y(y \in X \leftrightarrow F(y))$$

for any Σ^0_1 formula $F(y)$ and any Π^0_1 formula $G(y)$ with X not free in F.

WKL_0

We denote by WKL_0 the theory RCA_0 augmented with *Weak König's lemma*:

Every infinite binary tree has an infinite path.

ACA_0

We denote by ACA_0 the theory RCA_0 augmented with the comprehension scheme for arithmetic formulas.

ATR_0

We denote by ATR_0 the theory RCA_0 augmented with the *transfinite recursion scheme*:

For every arithmetic formula $\theta(x, X)$ and every well-ordered set $(A, <_A)$, the set $\bigoplus_{a \in A} Y_a$ exists, where Y_a is defined by

$$Y_a = \Theta(\bigoplus_{b <_A a} Y_b)$$

and $\Theta(X) = \{x \in \mathbb{N} : \theta(x, X)\}$.

Π^1_1-CA_0

We denote by Π^1_1-$\mathsf{CA}_0$the theory RCA_0 augmented with the comprehension scheme for Π^1_1 formulas.