{INDUCTION}

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INDUCTION HIERARCHY

I_{open}

I_{open} denotes the induction scheme :

$$(F(0) \wedge \forall x(F(x) \to F(x+1))) \to \forall x\, F(x)$$

restricted to quantifier free formulas.

$\mathrm{Q} + \mathrm{I}_{open}$ allows us to prove the most basic facts about $+$, $\times$ and $<$.

$\mathrm{I}\Delta_0^0$

$I\Delta_0^0$ denotes the induction scheme restricted to Δ_0^0 formulas.
$\mathrm{Q}+\mathrm{I}\Delta_0^0$ proves the basic properties of divisibility, and proves that substraction and Cantor's bijection are total functions.

$\mathrm{I}\Sigma_n^0$ and $\mathrm{I}\Pi_n^0$

$I\Sigma_n^0$ (resp. $I\Pi_n^0$) denotes the induction scheme restricted to Σ_n^0 (resp. Π_n^0) formulas.

We have $\mathrm{Q} \vdash \mathrm{I}\Sigma_n^0 \leftrightarrow \mathrm{I}\Pi_n^0$

$\mathrm{I}\Delta_n^0$

For $n > 0$, we denote by $\mathrm{I}\Delta_n^0$ the scheme :

$$\forall x(F(x) \leftrightarrow G(x)) \to (F(0) \wedge \forall x(F(x) \to F(x+1))) \to \forall x F(x)$$

with F Σ_n^0 and G Π_n^0.

MINIMUM SCHEME

$\mathrm{L}\Sigma_n^0$ and $\mathrm{L}\Pi_n^0$

We call *minimum scheme* for a formula $F(x)$ the statement :

$$\exists x\, F(x) \to \exists x(F(x) \wedge \forall y < x\, \neg F(y))$$

For $n > 0$, we denote by $L\Sigma_n^0$ (resp. $L\Pi_n^0$) the minimum scheme restricted to Σ_n^0 (resp. Π_n^0) formulas.

Minimum and induction

We have $\mathrm{Q} \vdash \mathrm{I}\Sigma_n^0 \leftrightarrow \mathrm{L}\Pi_n^0$ and $\mathrm{Q} \vdash \mathrm{I}\Pi_n^0 \leftrightarrow \mathrm{L}\Sigma_n^0$.
This gives us $\mathrm{Q} \vdash \mathrm{L}\Sigma_n^0 \leftrightarrow \mathrm{L}\Pi_n^0$.

BOUNDED COLLECTION

$\mathrm{B}\Sigma_n^0$ and $\mathrm{B}\Pi_n^0$

We call *bounded collection scheme* for a formula $F(x,y)$ the statement :

$$\forall n((\forall x < n\, \exists y F(x,y)) \to \exists b\, \forall x < n\, \exists y < b\, F(x,y))$$

For $n > 0$, we denote by $\mathrm{B}\Sigma_n^0$ (resp. $\mathrm{B}\Pi_n^0$) the bounded collection scheme restricted to Σ_n^0 (resp. Π_n^0) formulas.

We have $\mathrm{Q}+\mathrm{I}\Delta_0^0 \vdash \mathrm{B}\Pi_n^0 \leftrightarrow \mathrm{B}\Sigma_{n+1}^0$

Closure by bounded quantification of Σ_n^0/Π_n^0

$\mathrm{Q}+\mathrm{I}\Delta_0^0+\mathrm{B}\Sigma_n^0$, as well as $\mathrm{Q}+\mathrm{I}\Delta_0^0+\mathrm{B}\Pi_n^0$ proves the stability of Σ_n^0 (resp. Π_n^0) by $\wedge$, $\vee$, bounded quantification and $\exists$ (resp. $\forall$).

Bounded collection and induction

We have :

- $\mathrm{Q} \vdash \mathrm{I}\Sigma_n^0 \to \mathrm{B}\Sigma_n^0$
- $\mathrm{Q}+\mathrm{I}\Delta_0^0 \vdash \mathrm{B}\Sigma_{n+1}^0 \to \mathrm{I}\Sigma_n^0$
- $\mathrm{Q}+\mathrm{I}\Delta_0^0 \vdash \mathrm{B}\Sigma_n^0 \to \mathrm{I}\Delta_n^0$
- $\mathrm{Q}+\mathrm{I}\Sigma_1^0 \vdash \mathrm{B}\Sigma_n^0 \leftrightarrow \mathrm{I}\Delta_n^0$

FINITE / INFINITE SETS

Bounded / Unbounded predicates

A predicate $P(x)$ is said to be *bounded* (or *finite*) if there exists an x such that for all $y > x$, $\neg P(y)$. Otherwise, $P(x)$ is said to be *unbounded* (or *infinite*).
A set A is *finite* if its ownership predicate is finite. Otherwise, it is *infinite*.

Coding of finite sets

It is possible to define a *canonical code* for all finite sets in RCA_0 :
There exists a Δ_0^0 formula $\phi(x,y)$, which we will write "$x \in y$" such that RCA_0 proves the following properties :

- Every y codes a unique finite set A : the set of all x satisfying $x \in y$. We say that y is the *canonical code* of A.
- Every finite set has a canonical code.

BOUNDED COMPREHENSION

$\mathrm{BC}\Sigma_n^0$ and $\mathrm{BC}\Pi_n^0$

We call *bounded comprehension scheme* for a formula $F(x)$ the statement :

$$\forall t\, \exists n\, \forall y(y \in n \leftrightarrow (y < t \wedge F(y)))$$

For $n > 0$, we denote by $BC\Sigma_n^0$ (resp. $BC\Pi_n^0$) the bounded comprehension scheme restricted to Σ_n^0 (resp. Π_n^0) formulas.
We have : $\mathrm{RCA}_0 \vdash \mathrm{BC}\Sigma_n^0 \leftrightarrow \mathrm{BC}\Pi_n^0$.

$\mathrm{BC}\Delta_n^0$

For $n > 0$ we denote by $BC\Delta_n^0$ the scheme :

$$\forall x(F(x) \leftrightarrow G(x)) \to (\forall t\, \exists n\, \forall y(y \in n \leftrightarrow (y < t \wedge F(y))))$$

with F Σ_n^0 and G Π_n^0.

Bounded comprehension and induction

We have :

- $\mathrm{RCA}_0 \vdash \mathrm{BC}\Sigma_n^0 \leftrightarrow \mathrm{I}\Sigma_n^0$
- $\mathrm{RCA}_0 \vdash \mathrm{BC}\Delta_n^0 \leftrightarrow \mathrm{I}\Delta_n^0$

PROVABLY TOTAL FUNCTIONS

Provably functional

A formula $F(x,y)$ is RCA_0*-provably functional* if

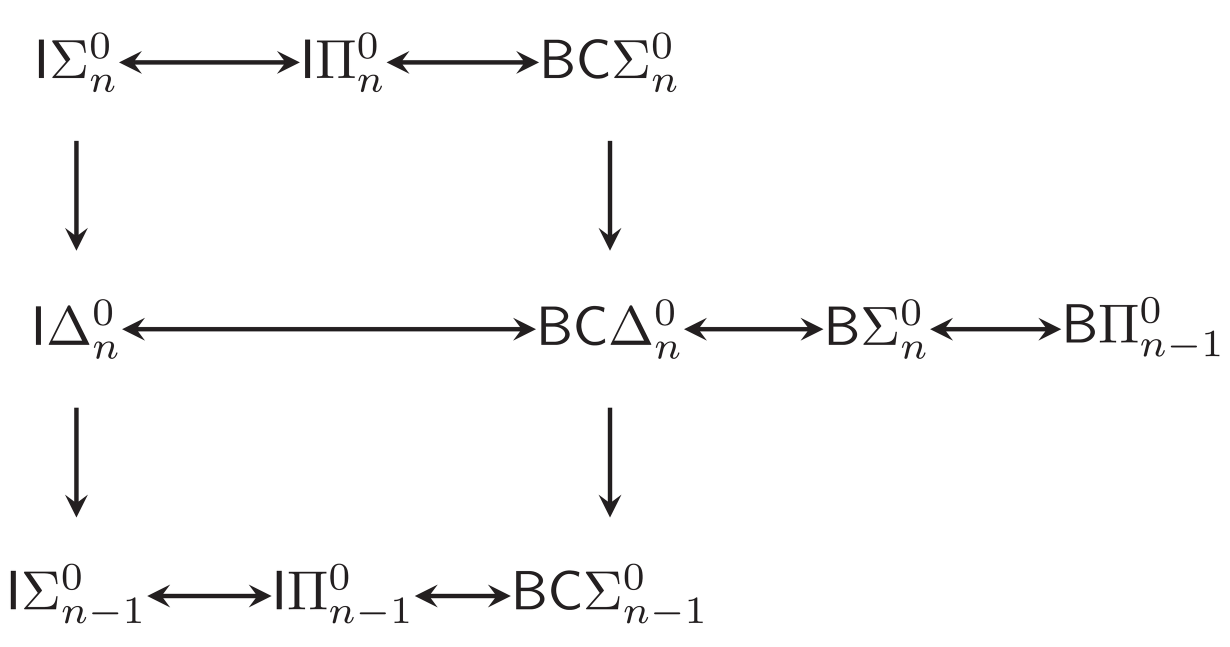
$$\mathrm{RCA}_0 \vdash \forall x \exists! y F(x,y)$$

A function $f : \omega \to \omega$ is RCA_0*-provably computable* if there is a RCA_0-provably functional Σ_1^0 formula $F(x,y)$ such that $F(n, f(n))$ holds for every $n \in \omega$.

Characterization

The RCA_0-provably computable functions are exactly the primitive recursive functions.

SUMMARY



Induction, collection and bounded comprehension hierarchies. Arrows denote implications over RCA_0.

CUTS

Cut

In a non-standard model M, a *cut* is a proper initial segment of M, non-empty and closed by successor.

Cuts and induction

For a second order structure (M,S), we have :

$$(M,S) \models \neg\mathrm{I}\Sigma_n^0 \Leftrightarrow \text{there is a } \Sigma_n^0 \text{ formula } F \text{ such that} \{x : F(x)\} \text{ is a cut}$$

If (M,S) is an ω-model, we have for all n :

$$(M,S) \models \mathrm{I}\Sigma_n^0$$