Rainbow Ramsey theorem for pairs

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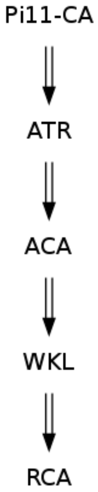
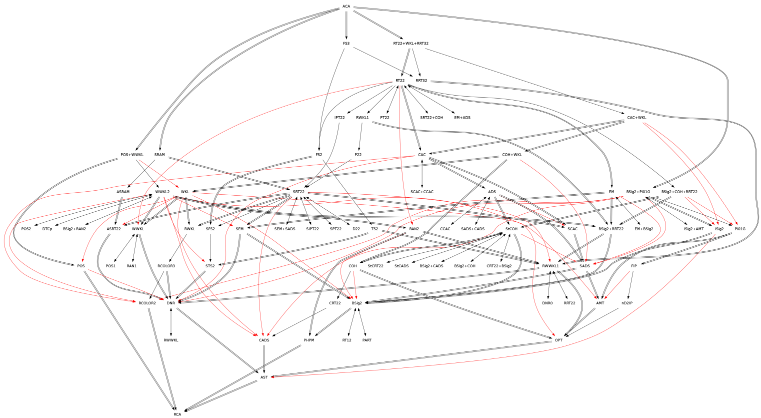
Summary

Plan

The "big five" subsystems



Reverse mathematics zoo



Plan

Ramsey theorems

$\mathbf{RT^n_k}$ (Ramsey theorem for n-tuples)

For every coloring function $f : \mathbb{N}^n \to \{0, .., k\}$ there is an infinite set H such that $f \upharpoonright H^n$ is monochromatic.

RT (Ramsey theorem)

$(\forall n)(\forall k)\mathbf{RT^n_k}$

Ramsey's Theorems

Theorem (Simpson)

(i) *For each* $n \geq 3$ *and* $k \geq 2$, $\mathbf{RCA_0} \vdash \mathbf{RT^n_k} \leftrightarrow \mathbf{ACA_0}$.

(ii) $\mathbf{RT}$ *is not provable in* $\mathbf{ACA_0}$.

Theorem

- $\mathbf{RCA_0} \vdash \mathbf{RT^1_k}$
- $\mathbf{RCA_0} \vdash \mathbf{ACA_0} \Rightarrow \mathbf{RT^2_2}$ *(1995)*
- $\mathbf{RT^2_2} \not\leftarrow \mathbf{WKL_0}$ *(2001)*
- $\mathbf{RT^2_2} \not\rightarrow \mathbf{WKL_0}$ *(2011)*

Stable Ramsey Theorem

Definition (Stable function)

A function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is stable if for some c

$$(\forall x)(\forall^\infty y)(f(x, y) = c)$$

$\mathbf{SRT^n_k}$ (Stable Ramsey theorem)

For every stable coloring function $f : \mathbb{N}^n \to \{0, .., k\}$ there is an infinite set H such that $f \restriction H^n$ is monochromatic.

Rainbow Ramsey Theorem

Definition (k-bounded function)

A coloring function $\mathbb{N}^n \to \mathbb{N}$ is k-bounded if $\texttt{card}\,\{x \in \mathbb{N}^n : f(x) = c\} \leq k$ for every color c.

$\mathbf{RRT^n_k}$ (Rainbow Ramsey Theorem)

For every k-bounded coloring function $f : \mathbb{N}^n \to \mathbb{N}$ there is an infinite set H such that $f \restriction H^n$ is injective.

Theorem (Galvin)

$\mathbf{RCA_0} \vdash \mathbf{RT^2_2} \to \mathbf{RRT^2_2}$

Diagonally Non-Computable function

Definition (Diagonally Non-Computable function)

A function f is DNC if $(\forall e)(f(e) \neq \Phi_e(e))$

DNC

For every set X there is a function **DNC** relative to X.

DNC[$\mathbf{0}'$]

For every set X there is a function **DNC** relative to X'.

Rainbow Ramsey Theorem

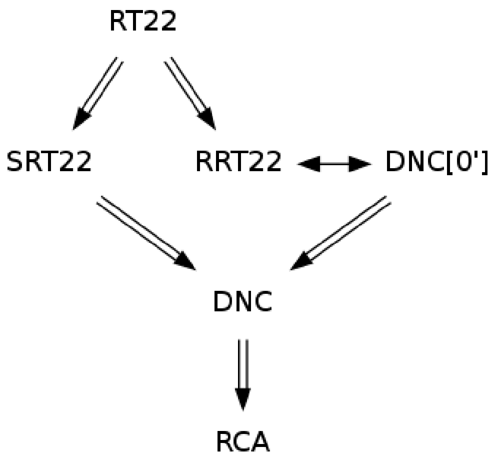
Theorem (Miller)

$\mathbf{RCA_0} \vdash \mathbf{DNC}[\mathbf{0}'] \leftrightarrow \mathbf{RRT^2_2}$

Theorem (Hirschfeldt & al.)

$\mathbf{RCA_0} \vdash \mathbf{SRT^2_2} \rightarrow \mathbf{DNC}$

Over $\mathbf{RCA_0}$...



How do $\mathbf{RRT^2_2}$ and $\mathbf{SRT^2_2}$ relate over $\mathbf{RCA_0}$?

Relating $\mathbf{RRT^2_2}$ to $\mathbf{SRT^2_2}$

Theorem (Bienvenu, Patey & Shafer)

There is an ω-model of $\mathbf{RRT^2_2}$ not model of $\mathbf{SRT^2_2}$.

Note: *In fact principles much weaker than $\mathbf{SRT^2_2}$ aren't implied by $\mathbf{RRT^2_2}$.*

Relating $\mathbf{RRT^2_2}$ to $\mathbf{SRT^2_2}$

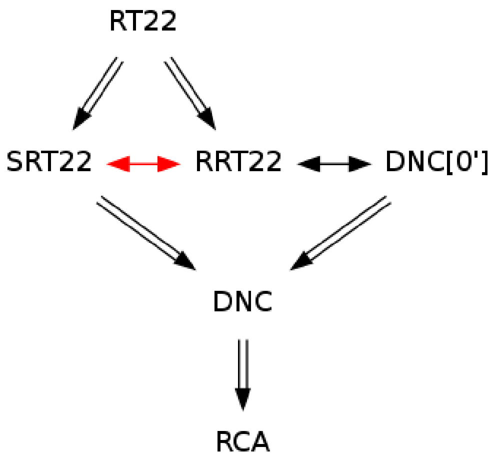
Theorem (Chong, Slaman, Yang)

There exists a non-standard model of $\mathbf{SRT^2_2}$ *with only* Δ^0_2 *(in fact low) sets.*

Corollary

$\mathbf{RCA_0} \nvdash \mathbf{SRT^2_2} \rightarrow \mathbf{DNC}[\mathbf{0}']$

Over $\mathbf{RCA_0}$...



Plan

Erdös-Moser Theorem

Definition (Tournament)

A tournament is a set $T \subseteq \mathbb{N} \times \mathbb{N}$ such that

$$(x, y) \in T \leftrightarrow (y, x) \notin T$$

Definition (Transitive tournament)

A tournament T is transitive if

$$(x, y) \in T \wedge (y, z) \in T \rightarrow (x, z) \in T$$

Definition (Stable tournament)

A tournament T is stable if

$$(\forall x)[(\forall^{\infty} y)((x, y) \in T) \vee (\forall^{\infty} y)((x, y) \notin T)]$$

Erdös-Moser Theorem

EM (Erdös-Moser Theorem)

Every infinite tournament has an infinite transitive subtournament.

SEM (Stable Erdös-Moser Theorem)

Every stable infinite tournament has an infinite transitive subtournament.

Erdös-Moser Theorem

Theorem (Bovykin and Weiermann)

- $\mathbf{RCA_0} \vdash \mathbf{RT^2_2} \leftrightarrow \mathbf{EM} + \mathbf{ADS}$
- $\mathbf{RCA_0} \vdash \mathbf{SRT^2_2} \leftrightarrow \mathbf{SEM} + \mathbf{SADS}$

Theorem (Bienvenu, Patey & Shafer)

There is an ω-model of $\mathbf{RRT^2_2}$ *not model of* $\mathbf{SEM}$.

Erdös-Moser Theorem

Theorem (Bienvenu, Patey & Shafer)

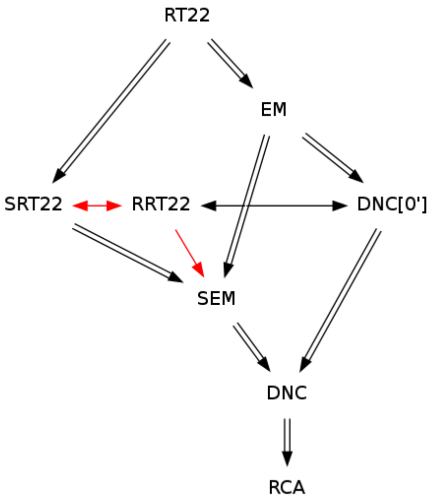
- $\mathbf{RCA_0} \vdash \mathbf{EM} \Rightarrow \mathbf{DNC}[\mathbf{0'}]$
- $\mathbf{RCA_0} \vdash \mathbf{SEM} \Rightarrow \mathbf{DNC}$

Idea: *Diagonalize (modulo encoding) against finite* $0'$*-c.e. sets using tournaments (respectively finite c.e. sets using stable tournaments).*

Question

Is there a direct proof of $\mathbf{RCA_0} \vdash \mathbf{EM} \rightarrow \mathbf{RRT^2_2}$ *?*

Over $\mathbf{RCA_0}$...



Plan

Thin Set

Definition (Thin Set)

TS(k) : For every coloring function $f : \mathbb{N}^k \to \mathbb{N}$ there exist an infinite set H such that $f(H^k) \neq \mathbb{N}$.

Definition (Stable Thin Set)

STS(k) : For every stable coloring function $f : \mathbb{N}^k \to \mathbb{N}$ there exist an infinite set H such that $f(H^k) \neq \mathbb{N}$.

Thin Set

Theorem (Cholak & al.)

$\mathbf{RCA_0} \vdash \mathbf{RT^2_2} \to \mathbf{TS(2)}$

Theorem (Bienvenu, Patey, Shafer)

$\mathbf{RCA_0} \vdash \mathbf{SRT^2_2} \to \mathbf{STS(2)}$

Theorem (Rice)

$\mathbf{RCA_0} \vdash \mathbf{STS(2)} \to \mathbf{DNC}$

Thin Set

Theorem (Bienvenu, Patey, Shafer)

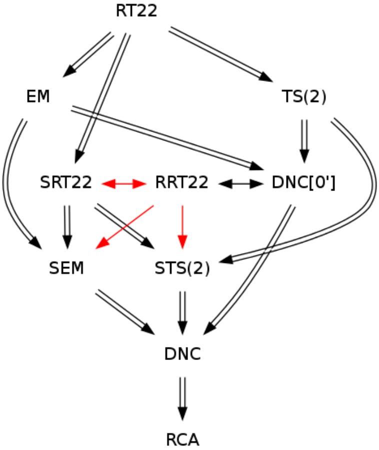
$\mathbf{RCA_0} \vdash \mathbf{TS(2)} \rightarrow \mathbf{DNC[0']}$

Theorem (Bienvenu, Patey, Shafer)

There is an ω*-model of* $\mathbf{RRT^2_2}$ *not model of* $\mathbf{STS(2)}$.

Idea: *Creating an instance whose class of solutions is almost surely non-computed by an oracle.*

Over $\mathbf{RCA_0}$...



Plan

References

Klaus Ambos-Spies, Bjørn Kjos-Hanssen, Steffen Lempp, and Theodore A Slaman.
Comparing DNR and WWKL.
Journal of Symbolic Logic, pages 1089–1104, 2004.

Jeremy Avigad, Edward T Dean, and Jason Rute.
Algorithmic randomness, reverse mathematics, and the dominated convergence theorem.
Annals of Pure and Applied Logic, 2012.

CT Chong, Theodore A Slaman, and Yue Yang.
The metamathematics of stable Ramsey's theorem for pairs.
To appear.

Stephen Flood.
Reverse mathematics and a Ramsey-type König's Lemma.
2011.

Jiayi Liu.
RT22 does not imply WKL0.
Journal of Symbolic Logic, 77(2):609–620, 2012.

Henry Towsner Manuel Lerman, Reed Solomon.
Separating principles below Ramsey's Theorem for Pairs.
2013.

Stephen George Simpson and Stephen G Simpson.
Subsystems of second order arithmetic, volume 42.
Springer Berlin, 1999.

Questions

Thank you for listening !