Reverse mathematics: Classifying principles by the no randomized algorithm property.

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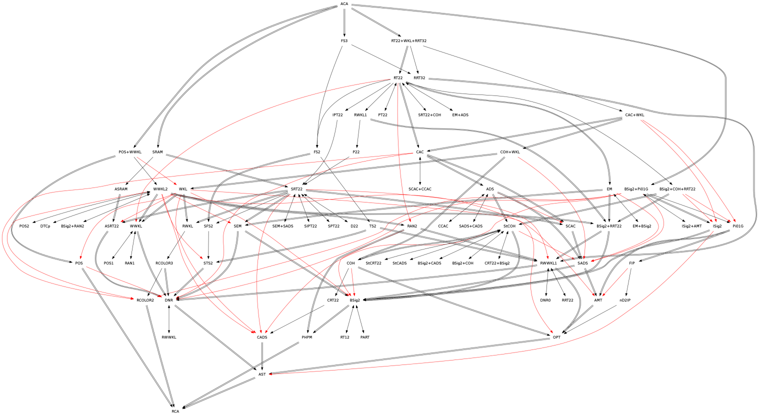
Summary

Plan

The “Big Five” subsystems

Reverse mathematics zoo



ω-structure

Definition (ω-structure)

$$\mathcal{M}_S = (\omega, S, +_\omega, \times_\omega, <_\omega)$$

Example (Minimal ω-model of RCA_0)

$COMP$ is the ω-structure where

$$S = \{X \in 2^\omega : X \text{ is computable}\}$$

Plan

No randomized algorithm property

Definition

Let $\vec{X}_i$ be a sequence of sets. $COMP(\vec{X}_i)$ is the ω-structure where

$$S = \bigcup_{i \in \omega} \{Y : Y \leq_T X_0 \oplus \cdots \oplus X_i\}.$$

Question

Fix a system P *and pick a sequence* $\vec{X}_i$ *at random.*
What is the probability that $COMP(\vec{X}_i) \models \mathrm{P}$ *?*

No randomized algorithm property

Definition

A system P has the *no randomized algorithm property* if when picking a sequence of sets $\vec{X}_i$, the probability that $COMP(\vec{X}_i) \models$ P is null.

Question

Which systems have the NRA property ?

No randomized algorithm property

Why *no randomized algorithm property ?*

- Consider a principle $\mathrm{P} = \forall Y \exists Z \Phi(Y, Z)$.
- If P has the NRA property, then for almost every sequence $\vec{X}_i$ there is a $Y \in COMP(\vec{X}_i)$ such that no probabilistic algorithm computes a Z such that $\Phi(Y, Z)$.

No randomized algorithm property

n-RAN (n-randomness)

For every X, there is a set Y which is n-random relative to X.

n-WWKL (n-weak weak König's lemma)

Every subtree of $2^{<\omega}$ of positive measure computable in $\emptyset^{(n-1)}$ has an infinite path.

Theorem (Avigdad, Dean & Rute)

For every standard n,

$$\mathrm{RCA_0} + \mathrm{B}\Sigma_\mathrm{n} \vdash \text{n-RAN} \leftrightarrow \text{n-WWKL}$$

No randomized algorithm property

Theorem

If a system S *has the NRA property*

$$\forall n \quad \mathrm{RCA}_0 \nvdash \text{n-WWKL} \rightarrow \mathrm{S}$$

Proof.

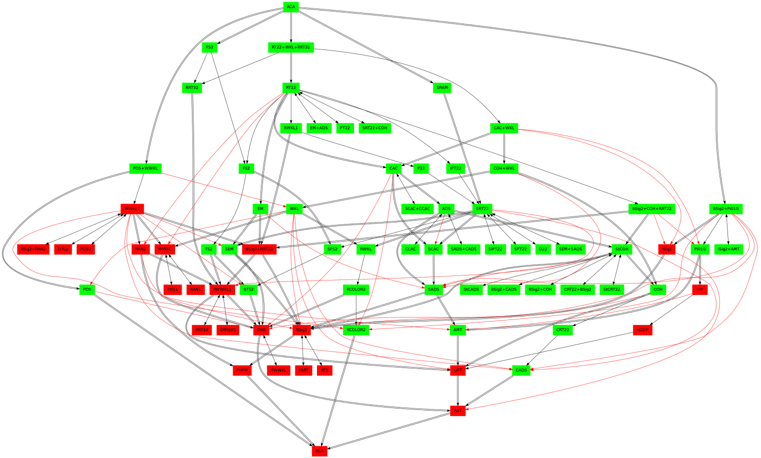
Pick the $\vec{X}_i$ at random. With probability 1, for all i, X_{i+1} is n-random relative to the join of the X_k, $k < i$. Therefore, with probability 1, $COMP(\vec{X}_i)$ is a model of n-WWKL. □

Plan

No randomized algorithm property

Which systems have the NRA property ?

Reverse mathematics zoo



Ordering

SADS (Stable ascending descending sequence)

Every linear order of order type $\omega + \omega^*$ has an infinite suborder of order type ω or ω^*.

Theorem (Csima & Mileti)

SADS *has the NRA property*

Proof.

There is a computable linear order of order type $\omega + \omega^*$ such that the measure of oracles computing an infinite suborder of order type ω or ω^* is null. $\square$

Ordering

CADS (Cohesive ascending descending sequence)

Every linear order has a suborder of order type $\omega + \omega^*$ or ω or ω^*.

Theorem (Bienvenu, Patey & Shafer)

CADS *has the NRA property*

Proof.

There is a computable linear order such that the measure of oracles computing an infinite suborder of order type $\omega + \omega^*$ or ω or ω^* is null. $\square$

Genericity

Π^0_1G (Π^0_1 genericity)

Any uniformly Π^0_1 collection of dense sets $D_i \subseteq 2^{<\omega}$ has a G such that $\forall i \exists s (G \upharpoonright s \in D_i)$.

Theorem (Kurtz)

The upward closure of the weakly 2-generic degrees has measure 0.

Theorem (Bienvenu, Patey & Shafer)

Π^0_1G *has the NRA property*

Genericity

1-GEN (1-genericity)

For any set X, there exists a set 1-generic relative to X.

Theorem (Kurtz)

Almost every set computes a 1-generic set.

Corollary

1-GEN *does not have the NRA property.*

Plan

Introduction

NRA property

Classification

Conclusion

Conclusion

- The following principles have the NRA property: Π^0_1G, CADS, SEM, RRT^3_2, POS, STS(2) $RCOLOR_2$.
- Any principle below n-WWKL for some n does not have the NRA property.
- It suffices to classify the whole zoo.

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Questions

Thank you for listening !