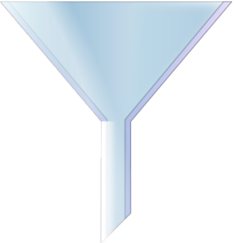
Lowness and avoidance

A guide to separation

Ludovic PATEY

Reverse mathematics

Infinitary mathematics



PRA

Theorem

$$T$$

Axioms Theorem

$$A_1, \ldots, A_n \Rightarrow T$$

Axioms Theorem

$$A_1, \ldots, A_n \Leftarrow T$$

Second-order arithmetics

$$t ::= 0 \mid 1 \mid x \mid t_1 + t_2 \mid t_1 \cdot t_2$$

$$\begin{aligned} f ::= \;\; & t_1 = t_2 \mid t_1 < t_2 \mid t_1 \in X \mid f_1 \vee f_2 \\ & \mid \neg f \mid \forall x.f \mid \exists x.f \mid \forall X.f \mid \exists X.f \end{aligned}$$

(Hilbert and Bernays)

Robinson's arithmetics

1. $m + 0 = m$
2. $m + (n + 1) = (m + n) + 1$
3. $m \times 0 = 0$
4. $m \times (n + 1) = (m \times n) + m$
5. $m + 1 \neq 0$
6. $m + 1 = n + 1 \to m = n$
7. $\neg(m < 0)$
8. $m < n + 1 \leftrightarrow (m < n \vee m = n)$

Comprehension scheme

$$\exists X \forall n (n \in X \Leftrightarrow \varphi(n))$$

for every formula $\varphi(n)$ where X appears freely.

Arithmetic hierarchy

$$\Sigma_n^0 \qquad \varphi(y) \equiv \exists x_1 \forall x_2 \ldots Q x_n \; \psi(y, x_1, \ldots, x_n)$$

$$\Pi_n^0 \qquad \varphi(y) \equiv \forall x_1 \exists x_2 \ldots Q x_n \; \psi(y, x_1, \ldots, x_n)$$

where ψ contains only bounded first-order quantifiers

A set is Γ if it is Γ-definable
A set is Δ_n^0 if it is Σ_n^0 and Π_n^0.

Computability $\equiv$ Definability

Theorem (Gödel)

A set is c.e. iff it is Σ^0_1 and computable iff it is Δ^0_1.

Theorem (Post)

A set is $\emptyset^{(n)}$-c.e. iff it is Σ^0_{n+1} and $\emptyset^{(n)}$-computable iff it is Δ^0_{n+1}.

Δ^0_1 comprehension scheme

$$\forall n(\varphi(n) \Leftrightarrow \psi(n)) \Rightarrow \exists X \forall n(n \in X \Leftrightarrow \varphi(n))$$

where $\varphi(n)$ is a Σ^0_1 formula where X does not occur freely, and ψ is a Π^0_1 formula.

Induction scheme

$$\varphi(0) \wedge \forall n(\varphi(n) \Rightarrow \varphi(n+1)) \Rightarrow \forall n \varphi(n)$$

for every formula $\varphi(n)$

Σ^0_1 induction scheme

$$\varphi(0) \wedge \forall n(\varphi(n) \Rightarrow \varphi(n+1)) \Rightarrow \forall n \varphi(n)$$

where $\varphi(n)$ is a Σ^0_1 formula

equivalent to

Σ^0_1 bounded comprehension scheme

$$\forall p \exists X \forall n(n \in X \Leftrightarrow n < p \wedge \varphi(n))$$

where $\varphi(n)$ is a Σ^0_1 formula where X does not occur freely

RCA_0

Robinson's arithmetics

$$m + 1 \neq 0$$
$$m + 1 = n + 1 \rightarrow m = n$$
$$\neg(m < 0)$$
$$m < n + 1 \leftrightarrow (m < n \vee m = n)$$

$$m + 0 = m$$
$$m + (n + 1) = (m + n) + 1$$
$$m \times 0 = 0$$
$$m \times (n + 1) = (m \times n) + m$$

Σ^0_1 induction scheme

$$\varphi(0) \wedge \forall n(\varphi(n) \Rightarrow \varphi(n+1))$$
$$\Rightarrow \forall n \varphi(n)$$

where $\varphi(n)$ is a Σ^0_1 formula

Δ^0_1 comprehension scheme

$$\forall n(\varphi(n) \Leftrightarrow \psi(n))$$
$$\Rightarrow \exists X \forall n(n \in X \Leftrightarrow \varphi(n))$$

where $\varphi(n)$ is a Σ^0_1 formula where X does not occur freely, and ψ is a Π^0_1 formula.

Reverse mathematics

Mathematics are computationally very structured

Almost every theorem is empirically equivalent to one among five big subsystems.

$$\Pi^1_1\mathrm{CA} \to \mathrm{ATR} \to \mathrm{ACA} \to \mathrm{WKL} \to \mathrm{RCA}_0$$

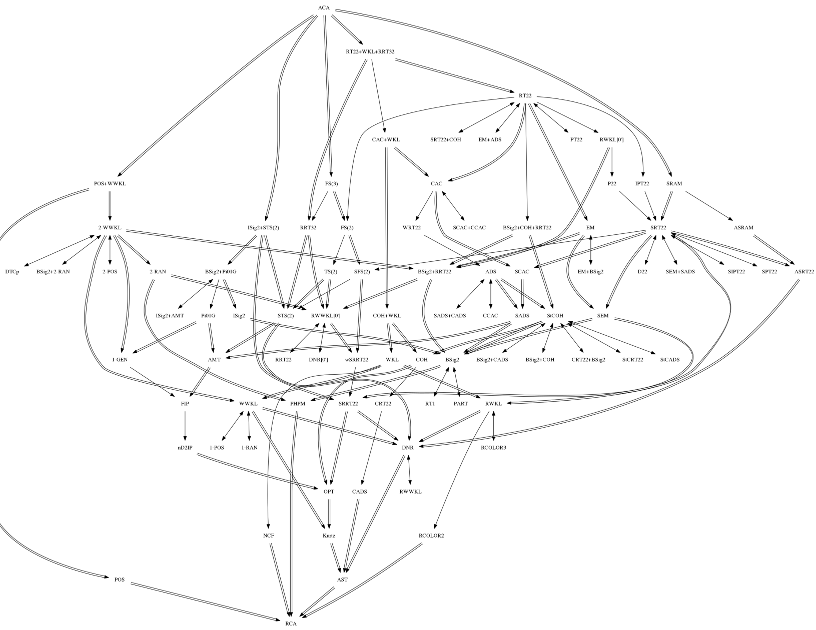
Reverse mathematics

Mathematics are computationally very structured

Almost every theorem is empirically equivalent to one among five big subsystems.

Except for Ramsey's theory...

$$
\begin{tikzcd}
 & \Pi^1_1\mathsf{CA} \arrow[d] \\
 & \mathsf{ATR} \arrow[d] \\
 & \mathsf{ACA} \arrow[d] \arrow[dl] \\
\mathsf{RT}^2_2 \arrow[dr] & \mathsf{WKL} \arrow[d] \\
 & \mathsf{RCA}_0
\end{tikzcd}
$$


ACA
RT22+WKL+RRT32
RT22
CAC+WKL
SRT22+COH
EM+ADS
PT22
RWKL[0']
POS+WWKL
FS(3)
CAC
P22
IPT22
SRAM
2-WWKL
ISig2+STS(2)
RRT32
FS(2)
WRT22
SCAC+CCAC
BSig2+COH+RRT22
EM
SRT22
ASRAM
DTCp
BSig2+2-RAN
2-POS
2-RAN
BSig2+Pi01G
TS(2)
SFS(2)
BSig2+RRT22
ADS
SCAC
EM+BSig2
D22
SEM+SADS
SIPT22
SPT22
ASRT22
ISig2+AMT
Pi01G
ISig2
STS(2)
RWWKL[0']
COH+WKL
SADS+CADS
CCAC
SADS
StCOH
SEM
1-GEN
AMT
RRT22
DNR[0']
wSRRT22
WKL
COH
BSig2
BSig2+CADS
BSig2+COH
CRT22+BSig2
StCRT22
StCADS
FIP
WWKL
PHPM
SRRT22
CRT22
RT1
PART
RWKL
nD2IP
1-POS
1-RAN
DNR
RCOLOR3
OPT
CADS
RWWKL
NCF
Kurtz
RCOLOR2
POS
AST
RCA

How to prove a

separation?

Given two statements P and Q.

How to prove that $\mathsf{RCA}_0 + \mathsf{P} \nvdash \mathsf{Q}$?

Build a model $\mathcal{M}$ such that

- $\mathcal{M} \models \mathsf{P}$
- $\mathcal{M} \not\models \mathsf{Q}$

ω-structure $\mathcal{M} = \{\omega, \mathcal{S}, <, +, \cdot\}$

(i) ω is the set of standard natural numbers

(ii) $<$ is the natural order

(iii) $+$ and $\cdot$ are the standard operations over natural numbers

(iv) $\mathcal{S} \subseteq \mathcal{P}(\omega)$

An ω-structure is fully specified by its second-order part $\mathcal{S}$.

Turing ideal $\mathcal{M}$

- $(\forall X \in \mathcal{M})(\forall Y \leq_T X)[Y \in \mathcal{M}]$
- $(\forall X, Y \in \mathcal{M})[X \oplus Y \in \mathcal{M}]$

Examples

- $\{X : X$ is computable $\}$
- $\{X : X \leq_T A \wedge X \leq_T B\}$ for some sets A and B

Let $\mathcal{M} = \{\omega, \mathcal{S}, <, +, \cdot\}$ be an ω-structure

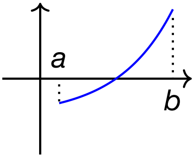
$$\mathcal{M} \models \mathsf{RCA_0}$$

$$\equiv$$

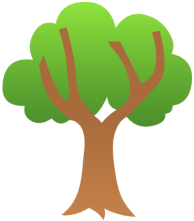
$\mathcal{S}$ is a Turing ideal

Many theorems can be seen as problems.

Intermediate value theorem
For every continuous function f over an interval $[a, b]$ such that $f(a) \cdot f(b) < 0$, there is a real $x \in [a, b]$ such that $f(x) = 0$.



König's lemma
Every infinite, finitely branching tree admits an infinite path.



Π^1_2-problem

$$\mathsf{P} \equiv \forall X[\varphi(X) \to \exists Y \psi(X, Y)]$$

where φ and ψ are arithmetic formulas

- P-instances: $\operatorname{dom} \mathsf{P} = \{X : \varphi(X)\}$
- P-solutions to X: $\mathsf{P}(X) = \{Y : \psi(X, Y)\}$

Given two Π^1_2-problems P and Q.

How to prove that $\mathsf{RCA}_0 + \mathsf{P} \nvdash \mathsf{Q}$?

Build a Turing ideal $\mathcal{M}$ such that

- $\mathcal{M} \models \mathsf{P}$
- $\mathcal{M} \not\models \mathsf{Q}$

Construct an ω-model of $\mathsf{RCA}_0 + \mathsf{P}$

Start with $\mathcal{M}_0 = \{Z : Z \leq_T \emptyset\}$

Given a Turing ideal $\mathcal{M}_n = \{Z : Z \leq_T U\}$ for some set U,

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1. pick an instance $X \in \mathcal{M}_n$ of P

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1. pick an instance $X \in \mathcal{M}_n$ of P
2. choose a solution Y to X
3. define $\mathcal{M}_{n+1} = \{Z : Z \leq_T Y \oplus U\}$

Construct an ω-model of $\mathsf{RCA}_0 + \mathsf{P}$

Start with $\mathcal{M}_0 = \{Z : Z \leq_T \emptyset\}$

Given a Turing ideal $\mathcal{M}_n = \{Z : Z \leq_T U\}$ for some set U,

1. pick an instance $X \in \mathcal{M}_n$ of P
2. choose a solution Y to X
3. define $\mathcal{M}_{n+1} = \{Z : Z \leq_T Y \oplus U\}$

Let $\mathcal{M} = \bigcup_n \mathcal{M}_n$. Then $\mathcal{M} \models \mathsf{RCA}_0 + \mathsf{P}$

Beware, adding sets to $\mathcal{M}$
may add solutions to instances of Q!

A weakness property is a collection of sets closed downward under the Turing reduction.

Exemples

- $\{X : X \text{ is low}\}$
- $\{X : A \not\leq_T X\}$ given a set A
- $\{X : X \text{ is hyperimmune-free}\}$

Let $\mathcal{W}$ be a weakness property.

A problem P preserves $\mathcal{W}$ if for every $Z \in \mathcal{W}$, every Z-computable instance X of P admits a solution Y such that $Y \oplus Z \in \mathcal{W}$

Lemma

If P preserves $\mathcal{W}$, then for every $Z \in \mathcal{W}$, there is an ω-model $\mathcal{M} \models \mathsf{RCA}_0 + \mathsf{P}$ with $Z \in \mathcal{M} \subseteq \mathcal{W}$.

Lemma

If P preserves $\mathcal{W}$ and Q does not, then $\mathsf{RCA}_0 + \mathsf{P} \nvdash \mathsf{Q}$

Cone avoidance

ACA_0

Arithmetic Comprehension Axiom

- Every increasing sequence of reals admits a supremum.
- Bolzano/Weierstrass theorem: Every sequence of reals admits a converging sub-sequence.
- Every countable commutative ring admits a maximal ideal.
- König's lemma: Every infinite, finitely branching tree admits an infinite path.
- Ramsey's theorem for colorings of $[\mathbb{N}]^3$.
- ...

ACA_0

Arithmetic Comprehension Axiom

$$X' = \{e : \exists t \ \Phi_e^X(e)[t] \downarrow\}$$

Lemma

$\mathsf{RCA}_0 \vdash \mathsf{ACA}_0 \leftrightarrow \forall X \, \exists Y \, (Y = X')$

Lemma

If a Π^1_2-problem P preserves $\mathcal{W}_{\emptyset'} = \{Z : \emptyset' \not\leq_T Z\}$, then $\mathsf{RCA}_0 + \mathsf{P} \nvdash \mathsf{ACA}_0$.

Cone avoidance

A Π^1_2-problem P admits cone avoidance if for every set Z, every set $C \not\leq_T Z$ and every Z-computable P-instance X, there is a P-solution Y to X such that $C \not\leq_T Y \oplus Z$.

P admits cone avoidance

$\equiv$

P preserves $\mathcal{W}_C = \{Z : C \not\leq_T Z\}$ for every set C

Strategy

Examples

Cohen forcing
Jockusch-Soare forcing

Pattern

Forcing question

Application

Pigeonhole forcing

Forcing in Computability Theory

Partial order
$(\mathbb{P}, \leq)$

Condition
$p \in \mathbb{P}$
approximation

Denotation
$[p] \subseteq 2^\omega$
class of candidates

Compatibility
If $q \leq p$ then $[q] \subseteq [p]$

Forcing in Computability Theory

Filter $\mathcal{F} \subseteq \mathbb{P}$

$\forall p \in \mathcal{F}\ \forall q \geq p\ q \in \mathcal{F}$

$\forall p, q \in \mathcal{F}, \exists r \in \mathcal{F}\ r \leq p, q$

Dense set $D \subseteq \mathbb{P}$

$\forall p \in \mathbb{P} \exists q \leq p\ q \in D$

Denotation

$[\mathcal{F}] = \bigcap_{p \in \mathcal{F}} [p]$

Forcing $p \Vdash \varphi(G)$

$\forall G \in [p]\ \varphi(G)$

Cohen forcing

$$(2^{<\omega}, \preceq)$$

$2^{<\omega}$ is the set of all finite binary strings

$\sigma \preceq \tau$ means σ is a prefix of τ

$[\sigma] = \{X \in 2^\omega : \sigma \prec X\}$

Theorem (Folklore)

Let $C \not\leq_T \emptyset$. For every sufficiently Cohen generic G, $C \not\leq_T G$.

Lemma

For every non-computable set C and Turing functional Φ_e, the following set is dense in $(2^{<\omega}, \preceq)$.

$$D = \{\sigma \in 2^{<\omega} : \sigma \Vdash \Phi_e^G \neq C\}$$

Given $\sigma \in 2^{<\omega}$, define the Σ^0_1 set

$$W = \{(x, v) : \exists \tau \succeq \sigma \ \Phi_e^\tau(x) \downarrow = v\}$$

- Case 1: $(x, 1 - C(x)) \in W$ for some x
  Then τ is an extension forcing $\Phi_e^G \neq C$
- Case 2: $(x, C(x)) \notin W$ for some x
  Then σ forces $\Phi_e^G \neq C$
- Case 3: W is a Σ^0_1 graph of C
  Impossible, since $C \not\leq_T \emptyset$

Weak König's lemma

$2^{<\omega}$ is the set of all finite binary strings

A binary tree is a set $T \subseteq 2^{<\omega}$ closed under prefixes

A path through T is an infinite sequence P such that every initial segment is in T

WKL Every infinite binary tree admits an infinite path.

Jockusch-Soare forcing

$$(\mathcal{T}, \subseteq)$$

$\mathcal{T}$ is the collection of infinite computable binary trees

$[T] = \{X \in 2^\omega : \forall \sigma \prec X\ \sigma \in T\}$

Theorem (Jockusch-Soare)

Let $C \not\leq_T \emptyset$. For every infinite computable binary tree $T \subseteq 2^{<\omega}$, there is a path $P \in [T]$ such that $C \not\leq_T P$.

Lemma

For every non-computable set C and Turing functional Φ_e, the following set is dense in $(\mathcal{T}, \subseteq)$.

$$D = \{T \in \mathcal{T} : T \Vdash \Phi_e^G \neq C\}$$

Given $T \in \mathcal{T}$, define the Σ^0_1 set

$$W = \{(x, v) : \exists \ell \in \mathbb{N} \forall \sigma \in 2^\ell \cap T \; \Phi^\sigma_e(x) \downarrow = v\}$$

- Case 1: $(x, 1 - C(x)) \in W$ for some x
  Then T forces $\Phi^G_e \neq C$
- Case 2: $(x, C(x)) \notin W$ for some x
  Then $\{\sigma \in T : \neg(\Phi^\sigma_e(x) \downarrow = v)\}$ forces $\Phi^G_e \neq C$
- Case 3: W is a Σ^0_1 graph of C
  Impossible, since $C \not\leq_T \emptyset$

Forcing question

$$p \mathrel{?{\vdash}} \varphi(G)$$

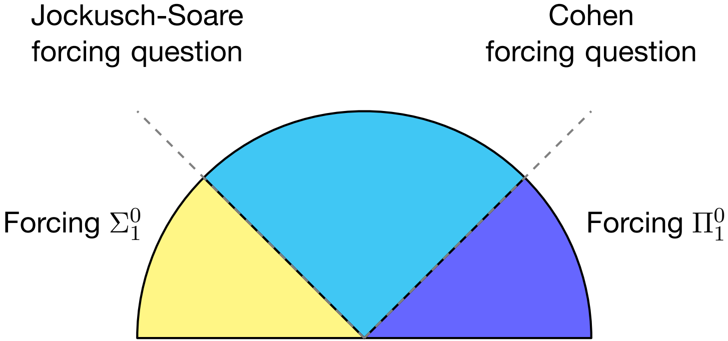
where $p \in \mathbb{P}$ and $\varphi(G)$ is Σ^0_1

Lemma

Let $p \in \mathbb{P}$ and $\varphi(G)$ be a Σ^0_1 formula.

(a) If $p \mathrel{?{\vdash}} \varphi(G)$, then $q \Vdash \varphi(G)$ for some $q \leq p$;

(b) If $p \mathrel{?{\nvdash}} \varphi(G)$, then $q \Vdash \neg\varphi(G)$ for some $q \leq p$.


Jockusch-Soare
forcing question
Cohen
forcing question
Forcing Σ^0_1
Forcing Π^0_1

Fix a notion of forcing $(\mathbb{P}, \leq)$.

A forcing question is Γ-preserving if for every $p \in \mathbb{P}$ and every Γ-formula $\varphi(G, x)$, the relation $p \mathrel{?\vdash} \varphi(G, x)$ is in Γ uniformly in x.

Lemma

Suppose $?\vdash$ is Σ^0_1-preserving. For every non-computable set C and Turing functional Φ_e, the following set is dense in $(\mathbb{P}, \leq)$.

$$D = \{p \in \mathbb{P} : p \Vdash \Phi_e^G \neq C\}$$

Given $p \in \mathbb{P}$, define the Σ^0_1 set

$$W = \{(x, v) : p \mathrel{?\vdash} \Phi^G_e(x) \downarrow = v\}$$

- Case 1: $(x, 1 - C(x)) \in W$ for some x
  Then there is an extension forcing $\Phi^G_e \neq C$
- Case 2: $(x, C(x)) \notin W$ for some x
  Then there is an extension forcing $\Phi^G_e \neq C$
- Case 3: W is a Σ^0_1 graph of C
  Impossible, since $C \not\leq_T \emptyset$

Pigeonhole principle

RT^1_k Every k-partition of $\mathbb{N}$ admits an infinite subset of a part.

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	....

$\longmapsto$

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	....

Theorem (Dzhafarov and Jockusch)

For every set $C \not\leq_T \emptyset$ and every 2-partition $A_0 \sqcup A_1 = \mathbb{N}$, there is some $i < 2$ and an infinite set $G \subseteq A_i$ such that $C \not\leq_T G$.

Theorem (Dzhafarov and Jockusch)

For every set $C \not\leq_T \emptyset$ and every 2-partition $A_0 \sqcup A_1 = \mathbb{N}$, there is some $i < 2$ and an infinite set $G \subseteq A_i$ such that $C \not\leq_T G$.

Input : a set $C \not\leq_T \emptyset$ and a 2-partition $A_0 \sqcup A_1 = \mathbb{N}$

Output : an infinite set $G \subseteq A_i$ such that $C \not\leq_T G$

$$(F_0, F_1, X)$$

Initial segment

Reservoir

- F_i is finite, X is infinite, $\max F_i < \min X$ (Mathias condition)
- $C \not\leq_T X$ (Weakness property)
- $F_i \subseteq A_i$ (Combinatorics)

Extension

$$(E_0, E_1, Y) \leq (F_0, F_1, X)$$

- $F_i \subseteq E_i$
- $Y \subseteq X$
- $E_i \setminus F_i \subseteq X$

Denotation

$$\langle G_0, G_1 \rangle \in [F_0, F_1, X]$$

- $F_i \subseteq G_i$
- $G_i \setminus F_i \subseteq X$

$$[E_0, E_1, Y] \subseteq [F_0, F_1, X]$$

$$(F_0, F_1, X) \Vdash \varphi(G_0, G_1)$$

Condition

Formula

$\varphi(G_0, G_1)$ holds for every $\langle G_0, G_1 \rangle \in [F_0, F_1, X]$

Input : a set $C \not\leq_T \emptyset$ and a 2-partition $A_0 \sqcup A_1 = \mathbb{N}$

Output : an infinite set $G \subseteq A_i$ such that $C \not\leq_T G$

Input : a set $C \not\leq_T \emptyset$ and a 2-partition $A_0 \sqcup A_1 = \mathbb{N}$

Output : an infinite set $G \subseteq A_i$ such that $C \not\leq_T G$

$$\Phi_{e_0}^{G_0} \neq C \vee \Phi_{e_1}^{G_1} \neq C$$

Input : a set $C \not\leq_T \emptyset$ and a 2-partition $A_0 \sqcup A_1 = \mathbb{N}$

Output : an infinite set $G \subseteq A_i$ such that $C \not\leq_T G$

$$\Phi_{e_0}^{G_0} \neq C \vee \Phi_{e_1}^{G_1} \neq C$$

The set $\{p \in \mathbb{P} : p \Vdash \Phi_{e_0}^{G_0} \neq C \vee \Phi_{e_1}^{G_1} \neq C\}$ is dense

Disjunctive forcing question

$$p \mathrel{?}\!\vdash \varphi_0(G_0) \vee \varphi_1(G_1)$$

where $p \in \mathbb{P}$ and $\varphi_0(G_0), \varphi_1(G_1)$ are Σ^0_1

Lemma

Let $p \in \mathbb{P}$ and $\varphi_0(G_0), \varphi_1(G_1)$ be Σ^0_1 formulas.

(a) If $p \mathrel{?}\!\vdash \varphi_0(G_0) \vee \varphi_1(G_1)$, then $q \Vdash \varphi_0(G_0) \vee \varphi_1(G_1)$ for some $q \leq p$;

(b) If $p \mathrel{?}\!\nvdash \varphi_0(G_0) \vee \varphi_1(G_1)$, then $q \Vdash \neg\varphi_0(G_0) \vee \neg\varphi_1(G_1)$ for some $q \leq p$.

Suppose the following relation is uniformly $\Sigma^0_1(X)$ whenever $\varphi_0(G_0), \varphi_1(G_1)$ are Σ^0_1

$$(F_0, F_1, X) \mathrel{?\vdash} \varphi_0(G_0) \vee \varphi_1(G_1)$$

Lemma

For every non-computable set C and Turing functionals Φ_{e_0}, Φ_{e_1}, the following set is dense in $(\mathbb{P}, \leq)$.

$$D = \{p \in \mathbb{P} : p \Vdash \Phi_{e_0}^{G_0} \neq C \vee \Phi_{e_1}^{G_1} \neq C\}$$

Consider the $\Sigma^0_1(X)$ set

$$W = \{(x, v) : p \mathrel{?\vdash} \Phi_{e_0}^{G_0}(x) \downarrow= v \vee \Phi_{e_0}^{G_0}(x) \downarrow= v\}$$

Problem: complexity of the instance

"Can we find an extension for this instance of RT^1_2?"

Definition

$$(F_0, F_1, X) \mathrel{?\vdash} \varphi_0(G_0) \vee \varphi_1(G_1)$$

$$\equiv$$

$$(\exists i < 2)(\exists E_i \subseteq X \cap A_i)\varphi_i(F_i \cup E_i)$$

The formula is $\Sigma^0_1(X \oplus A_i)$

Idea: make an overapproximation

"Can we find an extension for every instance of RT^1_2?"

Definition

$$(F_0, F_1, X) \mathrel{?\vdash} \varphi_0(G_0) \vee \varphi_1(G_1)$$

$$\equiv$$

$$(\forall B_0 \sqcup B_1 = \mathbb{N})(\exists i < 2)(\exists E_i \subseteq X \cap B_i)\varphi_i(F_i \cup E_i)$$

The formula is $\Sigma^0_1(X)$

Case 1: $p \mathrel{?}\!\vdash \varphi_0(G_0) \vee \varphi_1(G_1)$

Letting $B_i = A_i$, there is an extension $q \leq p$ forcing

$$\varphi_0(G_0) \vee \varphi_1(G_1)$$

Case 2: $p \mathrel{?}\!\nvdash \varphi_0(G_0) \vee \varphi_1(G_1)$

$$(\exists B_0 \sqcup B_1 = \mathbb{N})(\forall i < 2)(\forall E_i \subseteq X \cap B_i)\neg\varphi_i(F_i \cup E_i)$$

The condition $(F_0, F_1, X \cap B_i) \leq p$ forces

$$\neg\varphi_0(G_0) \vee \neg\varphi_1(G_1)$$

What we know so far...

Forcing question $?\vdash$	Notion of forcing $(\mathbb{P}, \leq)$
Σ^0_1-preserving	cone avoidance
$\dots$	$\dots$

Preservation of hyperimmunity

A function $g : \mathbb{N} \to \mathbb{N}$ dominates $f : \mathbb{N} \to \mathbb{N}$ if $\forall^\infty x\ g(x) \geq f(x)$.

A function $f : \mathbb{N} \to \mathbb{N}$ is a modulus for a set $A \subseteq \mathbb{N}$ if every function dominating f computes A.

A function $f : \mathbb{N} \to \mathbb{N}$ is hyperimmune if it is not dominated by any computable function.

An infinite set $A \subseteq \mathbb{N}$ is hyperimmune if there is no infinite computable sequence of pairwise disjoint blocs intersecting A.

Computation

Δ^1_1 (hyperarithmetic) sets

High degrees ($\mathbf{d}' \geq \mathbf{0}''$)

Hyperimmune sets

Function growth

Sets admitting a modulus

Functions dominating every computable function

Hyperimmune functions

A set G is weakly 1-generic if for every c.e. dense set of strings $W_e \subseteq 2^{<\mathbb{N}}$, there is some $\sigma \prec G$ in W_e.

Lemma

Every weakly 1-generic set is hyperimmune.

Given a computable sequence of pairwise disjoint blocs $(B_n)_{n\in\mathbb{N}}$ the following set is dense:

$$\{\sigma : \exists n\ |\sigma| > \max B_n \wedge B_n \cap \sigma = \emptyset\}$$

Lemma

Every hyperimmune function computes a weakly 1-generic set.

Given a hyperimmune function f, build an f-computable sequence $\sigma_0 \prec \sigma_1 \prec \dots$ Having defined σ_n, wait until time $f(|\sigma_n|)$ to see if some W_e enumerates an extension

(I cheat, slightly more complicated)

Preservation of hyperimmunity

A Π^1_2-problem P admits preservation of hyperimmunity if for every set Z, every Z-hyperimmune function f and every Z-computable P-instance X, there is a P-solution Y to X such that f is $Y \oplus Z$-hyperimmune.

P admits preservation of Z-hyperimmunity

$$\equiv$$

P preserves $\mathcal{W}_f = \{Z : f \text{ is } Z\text{-hyperimmune}\}$
for every function f

Cohen forcing

$$(2^{<\omega}, \preceq)$$

$2^{<\omega}$ is the set of all finite binary strings

$\sigma \preceq \tau$ means σ is a prefix of τ

$[\sigma] = \{X \in 2^{\omega} : \sigma \prec X\}$

Theorem (Folklore)

Let $f : \mathbb{N} \to \mathbb{N}$ be hyperimmune. For every sufficiently Cohen generic G, f is G-hyperimmune.

Lemma

For every hyperimmune function $f : \mathbb{N} \to \mathbb{N}$ and Turing functional Φ_e, the following set is dense in $(2^{<\omega}, \preceq)$.

$$D = \{\sigma \in 2^{<\omega} : \sigma \Vdash \exists x\ \Phi_e^G(x) \uparrow \vee \exists x\ \Phi_e^G(x) < f(x)\}$$

Given $\sigma \in 2^{<\omega}$, define the partial computable function:
$h(x) = y$ for the least y such that

$$\exists \tau \succeq \sigma \ \Phi_e^\tau(x) \downarrow = y$$

- Case 1: $h(x) < f(x)$ for some $x \in \operatorname{dom} h$.
  Then τ is an extension forcing $\Phi_e^G(x) < f(x)$
- Case 2: $x \notin \operatorname{dom} h$ for some x
  Then σ forces $\Phi_e^G(x) \uparrow$
- Case 3: h is total and dominates f.
  Impossible, since f is hyperimmune

Fix a notion of forcing $(\mathbb{P}, \leq)$.

A forcing question is Γ-compact if for every $p \in \mathbb{P}$ and every Γ-formula $\varphi(G, x)$, if $p \operatorname{?\vdash} \exists x\, \varphi(G, x)$ then there is a finite set $F \subseteq \mathbb{N}$ such that $p \operatorname{?\vdash} \exists x \in F\, \varphi(G, x)$.

Lemma

Suppose $\operatorname{?\vdash}$ is Σ^0_1-preserving and Σ^0_1-compact. For every hyperimmune function $f : \mathbb{N} \to \mathbb{N}$ and Turing functional Φ_e, the following set is dense in $(\mathbb{P}, \leq)$.

$$D = \{p \in \mathbb{P} : p \Vdash \exists x\, \Phi^G_e(x) \uparrow \vee \exists x\, \Phi^G_e(x) < f(x)\}$$

Given $p \in \mathbb{P}$, define the partial computable function:
$h(x) = 1 + \max F$ for the least F such that

$$p \mathrel{?\vdash} \exists y \in F\ \Phi_e^G(x) \downarrow = y$$

- Case 1: $h(x) < f(x)$ for some $x \in \operatorname{dom} h$.
  Then there is an extension forcing $\Phi_e^G(x) \leq \max F < f(x)$
- Case 2: $x \notin \operatorname{dom} h$ for some x
  Then $p \mathrel{?\nvdash} \exists y\ \Phi_e^G(x) \downarrow = y$. There is an extension forcing $\Phi_e^G(x) \uparrow$
- Case 3: h is total and dominates f.
  Impossible, since f is hyperimmune

Theorem

A Π^1_2-problem admits cone avoidance iff it admits preservation of hyperimmunity.

- If a problem admits cone avoidance, it can avoid ω cones simultaneously.
- There are problems which admit preservation of k hyperimmunities, but not $k+1$ simultaneously.

What we know so far...

Forcing question $?\vdash$	Notion of forcing $(\mathbb{P}, \leq)$
Σ^0_1-preserving	cone avoidance
Σ^0_1-preserving and Σ^0_1-compact	preservation of hyperimmunity
$\dots$	$\dots$

Compactness avoidance

WKL_0

Weak König's lemma

- Every infinite binary tree admits an infinite path
- Heine/Borel cover lemma: Every cover of the $[0, 1]$ interval by a sequence of open sets admits a finite sub-cover.
- Every real-valued function over $[0, 1]$ is bounded.
- Gödel's completeness theorem: every countable set of statements in predicate calculus admits a countable model.
- Every countable commutative ring admits a prime ideal.
- ...

A function $f : \mathbb{N} \to \mathbb{N}$ is diagonally non-computable (DNC) if

$$\forall e\ f(e) \neq \Phi_e(e)$$

Lemma

There exists a computable infinite binary tree $T \subseteq 2^{<\mathbb{N}}$ such that $[T]$ are the $\{0,1\}$-valued DNC functions.

- $T = \{\sigma \in 2^{<\mathbb{N}} : \forall e < |\sigma|\ \sigma(e) \neq \Phi_e(e)[|\sigma|]\}$.

Lemma

For every computable infinite binary tree T, every $\{0,1\}$-valued DNC function computes a path.

- Given $\sigma \in T$ and $x \in \mathbb{N}$, let Φ_{e_σ} explore the branches below $\sigma \cdot 0$ and $\sigma \cdot 1$.
- If the branch below $\sigma \cdot i$ is the first to die, then halt and output i.
- For every σ extensible in T, $\sigma \cdot f(e_\sigma)$ is extensible in T.

Cohen forcing

$$(2^{<\omega}, \preceq)$$

$2^{<\omega}$ is the set of all finite binary strings

$\sigma \preceq \tau$ means σ is a prefix of τ

$[\sigma] = \{X \in 2^{\omega} : \sigma \prec X\}$

Theorem (Folklore)

Every sufficiently Cohen generic G computes no $\{0, 1\}$-valued DNC function.

Lemma

For every $\{0, 1\}$-valued Turing functional Φ_e, the following set is dense in $(2^{<\omega}, \preceq)$.

$$D = \{\sigma \in 2^{<\omega} : \sigma \Vdash \exists x\ \Phi_e^G(x) \uparrow \vee \exists x\ \Phi_e^G(x) \downarrow = \Phi_x(x)\}$$

Given $\sigma \in 2^{<\omega}$, define the Σ^0_1 set

$$W = \{(x, v) : \exists \tau \succeq \sigma \ \Phi^\tau_e(x) \downarrow = v\}$$

- Case 1: $(x, \Phi_x(x)) \in W$ for some x such that $\Phi_x(x) \downarrow$
  Then τ is an extension forcing $\Phi^G_e(x) = \Phi_x(x)$
- Case 2: $(x, 0), (x, 1) \notin W$ for some x
  Then σ forces $\Phi^G_e(x) \uparrow$
- Case 3: W is a Σ^0_1 graph of a DNC function
  Impossible, since no DNC function is computable.

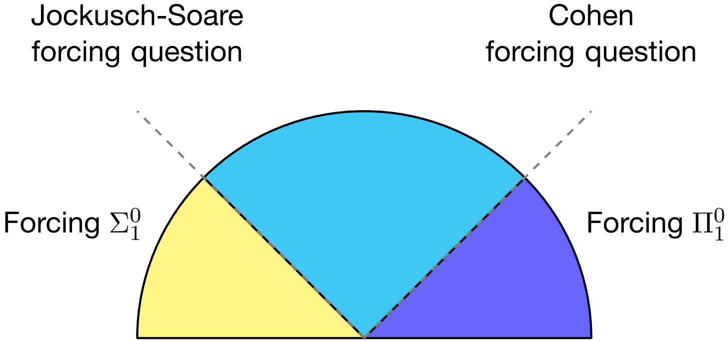
Fix a notion of forcing $(\mathbb{P}, \leq)$.

A forcing question is Π^0_n-merging if for every $p \in \mathbb{P}$ and every pair of Σ^0_n-formulas $\varphi(G), \psi(G)$ such that $p \mathrel{?\!\nvdash} \varphi(G)$ and $p \mathrel{?\!\nvdash} \psi(G)$, there is an extension $q \leq p$ such that $q \Vdash \neg\varphi(G) \wedge \neg\psi(G)$.

Lemma

Suppose $?\vdash$ is Σ^0_1-preserving and Π^0_1-merging. For every $\{0,1\}$-valued functional Φ_e, the following set is dense in $(\mathbb{P}, \leq)$.

$$D = \{p \in \mathbb{P} : p \Vdash \exists x\ \Phi_e^G(x) \uparrow \vee \exists x\ \Phi_e^G(x) \downarrow = \Phi_x(x)\}$$


Jockusch-Soare
forcing question
Cohen
forcing question
Forcing Σ^0_1
Forcing Π^0_1

Solovay forcing

$$(\mathcal{C}, \subseteq)$$

$\mathcal{C}$ is the collection of closed classes of positive measure in $2^{\mathbb{N}}$

Theorem

For every sufficiently Solovay generic G, G computes no $\{0,1\}$-valued DNC function.

Lemma

For every $\{0,1\}$-valued Turing functional Φ_e, the following set is dense in $\mathcal{C}$.

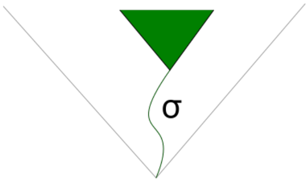
$$D = \{\mathcal{P} \in \mathcal{C} : \mathcal{P} \Vdash \exists x\ \Phi_e^G(x) \uparrow \vee \exists x\ \Phi_e^G(x) \downarrow = \Phi_x(x)\}$$

Lebesgue density lemma

Lemma

For every closed class $\mathcal{P} \subseteq 2^{\mathbb{N}}$ of positive measure and every $\epsilon > 0$, there is some $\sigma \in 2^{<\mathbb{N}}$ such that

$$\frac{\mu(\mathcal{P} \cap [\sigma])}{\mu([\sigma]) \geq 1 - \epsilon}$$



Given a closed class $\mathcal{P} \subseteq 2^{\mathbb{N}}$ and $\sigma \in 2^{<\mathbb{N}}$ such that $\mu(\mathcal{P}) \cap [\sigma]) > 0.9 \times \mu([\sigma])$, define the Σ^0_1 set

$$W = \{(x, v) : \mu(Z : \Phi_e^{\sigma \cdot Z}(x) \downarrow = v) > 0.2\}$$

- Case 1: $(x, \Phi_x(x)) \in W$ for some x such that $\Phi_x(x) \downarrow$
  Then pick $\tau \in 2^{<\mathbb{N}}$ such that $\mu(\mathcal{P} \cap [\tau]) > 0$ and $\Phi_e^{\tau}(x) \downarrow = \Phi_x(x)$.
  The class $\mathcal{P} \cap [\tau]$ is an extension forcing $\Phi_e^G(x) = \Phi_x(x)$

- Case 2: $(x, 0), (x, 1) \notin W$ for some x
  Then $\mathcal{P} \cap [\sigma] \cap \{Y : \Phi_e^Y(x) \uparrow\}$ forces $\Phi_e^G(x) \uparrow$

- Case 3: W is a Σ^0_1 graph of a DNC function
  Impossible, since no DNC function is computable.

DNC

Diagonal Non-Computability

- For every set X, there exists an X-DNC function f, that is, $\forall e, f(e) \neq \Phi_e^X(e)$.
- For every set X, there exists an X-fixpoint-free function f, that is, $\forall e, W_{f(e)}^X \neq W_e^X$.
- For every set X, there exists a function f such that $\forall n, C^X(f(n)) \geq n$.
- For every set X, there exists an infinite subset of an X-random set.
- RWWKL: For every binary tree of positive measure $T \subseteq 2^{<\mathbb{N}}$, there is an infinite homogeneous set.
- ...

Lemma

There is a probabilistic algorithm to compute a DNC function.

Algorithm	Probability of error
Pick $f(0)$ at random in $[0, 2^2]$	$\leq 2^{-2}$
Pick $f(1)$ at random in $[0, 2^3]$	$\leq 2^{-3}$
Pick $f(2)$ at random in $[0, 2^4]$	$\leq 2^{-4}$
$\ldots$	

Global probability of error: at most $\sum_n 2^{-n-2} = 0.5$.

Cohen forcing

$$(2^{<\omega}, \preceq)$$

$2^{<\omega}$ is the set of all finite binary strings

$\sigma \preceq \tau$ means σ is a prefix of τ

$[\sigma] = \{X \in 2^\omega : \sigma \prec X\}$

Theorem (Folklore)

Every sufficiently Cohen generic G computes no DNC function.

Lemma

For every Turing functional Φ_e, the following set is dense in $(2^{<\omega}, \preceq)$.

$$D = \{\sigma \in 2^{<\omega} : \sigma \Vdash \exists x\ \Phi_e^G(x) \uparrow \vee \exists x\ \Phi_e^G(x) \downarrow = \Phi_x(x)\}$$

Given $\sigma \in 2^{<\omega}$, define the Σ^0_1 set

$$W = \{(x, v) : \exists \tau \succeq \sigma \ \Phi^{\tau}_e(x) \downarrow = v\}$$

- Case 1: $(x, \Phi_x(x)) \in W$ for some x such that $\Phi_x(x) \downarrow$
  Then τ is an extension forcing $\Phi^G_e(x) = \Phi_x(x)$
- Case 2: $\exists x \, \forall y \, (x, y) \notin W$
  Then σ forces $\Phi^G_e(x) \uparrow$
- Case 3: W is a Σ^0_1 graph of a DNC function
  Impossible, since no DNC function is computable.

Fix a notion of forcing $(\mathbb{P}, \leq)$.

A forcing question is countably Π^0_n-merging if for every $p \in \mathbb{P}$ and every countable sequence of Σ^0_n-formulas $(\varphi_n(G))_{n\in\mathbb{N}}$ such that for every n, $p \mathrel{?}\!\nvdash \varphi_n(G)$, there is an extension $q \leq p$ such that for every n, $q \Vdash \neg\varphi_n(G)$.

Lemma

Suppose $?\vdash$ is Σ^0_1-preserving and countably Π^0_1-merging. For every Turing functional Φ_e, the following set is dense in $(\mathbb{P}, \leq)$.

$$D = \{p \in \mathbb{P} : p \Vdash \exists x\ \Phi^G_e(x) \uparrow \vee \exists x\ \Phi^G_e(x) \downarrow = \Phi_x(x)\}$$

What we know so far...

Forcing question $?\vdash$	Notion of forcing $(\mathbb{P}, \leq)$
Σ^0_1-preserving	cone avoidance
Σ^0_1-preserving and Σ^0_1-compact	preservation of hyperimmunity
Σ^0_1-preserving and Π^0_1-merging	PA avoidance
Σ^0_1-preserving and ω-Π^0_1-merging	DNC avoidance
$\dots$	$\dots$

Conservation theorems

Induction scheme

$$\varphi(0) \land \forall x(\varphi(x) \to \varphi(x+1)) \to \forall y \varphi(y)$$

for every formula $\varphi(x)$

Collection scheme

$$(\forall x < a)(\exists y)\varphi(x, y) \to (\exists b)(\forall x < a)(\exists y < b)\varphi(x, y)$$

for every $a \in \mathbb{N}$ and every formula $\varphi(x, y)$

Over $\mathsf{Q} + \mathsf{I}\Delta^0_0 + \mathsf{exp}$

Induction	Collection	Least principle	Regularity
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathsf{I}\Sigma^0_2 \equiv \mathsf{I}\Pi^0_2$		$\mathsf{L}\Pi^0_2 \equiv \mathsf{L}\Sigma^0_2$	Σ^0_2-regularity
$\mathsf{I}\Delta^0_2$	$\mathsf{B}\Sigma^0_2 \equiv \mathsf{B}\Pi^0_1$	$\mathsf{L}\Delta^0_2$	Δ^0_2-regularity
$\mathsf{I}\Sigma^0_1 \equiv \mathsf{I}\Pi^0_1$		$\mathsf{L}\Pi^0_1 \equiv \mathsf{L}\Sigma^0_1$	Σ^0_1-regularity
$\mathsf{I}\Delta^0_1$	$\mathsf{B}\Sigma^0_1 \equiv \mathsf{B}\Pi^0_0$	$\mathsf{L}\Delta^0_1$	Δ^0_1-regularity

- exp: totality of the exponential
- A set X is M-regular if every initial segment of X is M-coded
- Least principle: every non-empty set admits a minimum element

Over $\mathsf{Q} + \mathsf{I}\Delta^0_0 + \exp$

Induction	Collection	Least principle	Regularity
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathsf{I}\Sigma^0_2 \equiv \mathsf{I}\Pi^0_2$		$\mathsf{L}\Pi^0_2 \equiv \mathsf{L}\Sigma^0_2$	Σ^0_2-regularity
$\mathsf{I}\Delta^0_2$	$\mathsf{B}\Sigma^0_2 \equiv \mathsf{B}\Pi^0_1$	$\mathsf{L}\Delta^0_2$	Δ^0_2-regularity
$\mathsf{I}\Sigma^0_1 \equiv \mathsf{I}\Pi^0_1$		$\mathsf{L}\Pi^0_1 \equiv \mathsf{L}\Sigma^0_1$	Σ^0_1-regularity
$\mathsf{I}\Delta^0_1$	$\mathsf{B}\Sigma^0_1 \equiv \mathsf{B}\Pi^0_0$	$\mathsf{L}\Delta^0_1$	Δ^0_1-regularity

$$\mathsf{RCA}_0 \equiv \mathsf{Q} + \Delta^0_1\text{-comprehension} + \mathsf{I}\Sigma^0_1$$

Over $\mathsf{Q} + \mathsf{I}\Delta^0_0 + \mathsf{exp}$

Induction	Collection	Least principle	Regularity
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathsf{I}\Sigma^0_2 \equiv \mathsf{I}\Pi^0_2$		$\mathsf{L}\Pi^0_2 \equiv \mathsf{L}\Sigma^0_2$	Σ^0_2-regularity
$\mathsf{I}\Delta^0_2$	$\mathsf{B}\Sigma^0_2 \equiv \mathsf{B}\Pi^0_1$	$\mathsf{L}\Delta^0_2$	Δ^0_2-regularity
$\mathsf{I}\Sigma^0_1 \equiv \mathsf{I}\Pi^0_1$		$\mathsf{L}\Pi^0_1 \equiv \mathsf{L}\Sigma^0_1$	Σ^0_1-regularity
$\mathsf{I}\Delta^0_1$	$\mathsf{B}\Sigma^0_1 \equiv \mathsf{B}\Pi^0_0$	$\mathsf{L}\Delta^0_1$	Δ^0_1-regularity

$$\mathsf{RCA}^*_0 \equiv \mathsf{Q} + \Delta^0_1\text{-comprehension} + \mathsf{I}\Delta^0_0 + \mathsf{exp}$$

First-order part of T:

set of its first-order sentences

Induction	System	First-order part
$\vdots$	$\vdots$	$\vdots$
$\mathsf{I}\Sigma^0_2 \equiv \mathsf{I}\Pi^0_2$	$\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$	$\mathsf{Q} + \mathsf{I}\Sigma_2$
$\mathsf{I}\Delta^0_2$	$\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$	$\mathsf{Q} + \mathsf{I}\Delta_2$
$\mathsf{I}\Sigma^0_1 \equiv \mathsf{I}\Pi^0_1$	RCA_0	$\mathsf{Q} + \mathsf{I}\Sigma_1$
$\mathsf{I}\Delta^0_1 + \mathsf{exp}$	RCA^*_0	$\mathsf{Q} + \mathsf{I}\Delta_1 + \mathsf{exp}$

Fix a family of formulas Γ.

A theory T_1 is Γ-conservative over T_0 if every Γ-sentence provable over T_1 is provable over T_0.

If T_1 is a Π^1_1-conservative extension of T_0, then they have the same first-order part.

A second-order structure $\mathcal{N} = (N, T)$ is an ω-extension of $\mathcal{M} = (M, S)$ if $N = M$, $T \supseteq S$, $+^{\mathcal{N}} = +^{\mathcal{M}}$ and $<^{\mathcal{N}}=<^{\mathcal{M}}$.

Theorem

If every countable model of $\mathcal{M} \models T_0$ admits an ω-extension $\mathcal{N} \models T_1$, then T_1 is Π^1_1-conservative over T_0.

- Suppose $T_0 \nvdash \forall X \phi(X)$. Let $\mathcal{M} \models T_0 \wedge \exists X \neg \phi(X)$.
- Let $\mathcal{N} \models T_1$ be an ω-extension of $\mathcal{M}$.
- Then $\mathcal{N} \models T_1 \wedge \exists X \neg \phi(X)$. So $T_1 \nvdash \forall X \phi(X)$.

Let $\mathcal{M} = (M, S)$ be a second-order structure, and $G \subseteq M$. $\mathcal{M}[G]$ is the smallest ω-extension containing the $\Delta^0_1(\mathcal{M} \cup \{G\})$ sets.

Theorem

Let P be a Π^1_2-problem and T be a theory. If for every countable model $\mathcal{M} \models T$ and every $X \in \mathcal{M}$ such that $\mathcal{M} \models (X \in \operatorname{dom} \mathsf{P})$, there is a set $Y \subseteq M$ such that $\mathcal{M}[Y] \models T + (Y \in \mathsf{P}(X))$, then $T + \mathsf{P}$ is Π^1_1-conservative over T.

$$\mathcal{M} \subseteq \mathcal{M}[Y_0] \subseteq \mathcal{M}[Y_0][Y_1] \subseteq \dots$$

WKL_0

Weak König's lemma

- Every infinite binary tree admits an infinite path

Theorem (Harrington)

WKL_0 is Π^1_1-conservative over RCA_0

Theorem (Harrington)

Let $\mathcal{M} = (M, S) \models \mathrm{RCA}_0$ be a countable model and $T \subseteq 2^{<M}$ be an infinite tree in S. There is a path $G \in [T]$ such that $\mathcal{M}[G] \models \mathrm{RCA}_0$.

$$(\mathbb{P}, \leq)$$

The set of all infinite binary trees in S ordered by inclusion

$$T \;?\!\vdash \exists y \psi(y, G \upharpoonright_y)$$

there is some $\ell \in M$ such that for every $\sigma \in T$ of length ℓ, $\psi(y, \sigma \upharpoonright y)$ for some $y < \ell$.

$$T \mathrel{?\vdash} \exists y \psi(y, G \upharpoonright_y)$$

there is some $\ell \in M$ such that
for every $\sigma \in T$ of length ℓ,
$\psi(y, \sigma \upharpoonright y)$ for some $y < \ell$.

Lemma

Let T be a condition and $\varphi(G)$ be a $\Sigma^0_1(\mathcal{M})$-formula.

1. If $T \mathrel{?\vdash} \varphi(G)$ then T forces $\varphi(G)$
2. If $T \mathrel{?\nvdash} \varphi(G)$ then there is an extension $T_1 \subseteq T$ forcing $\neg\varphi(G)$

Lemma (Friedman)

Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0$ and $G \subseteq M$ be such that $\mathcal{M} \cup \{G\} \models \mathsf{I}\Sigma^0_1$. Then $\mathcal{M}[G] \models \mathsf{RCA}_0$.

Lemma

Let T be a condition and $\varphi(x, X)$ be a $\Sigma^0_1(\mathcal{M})$-formula such that T forces $\neg\varphi(b, G)$ for some $b \in M$. Then there is an extension $T_1 \subseteq T$ such that

- Either T_1 forces $\neg\varphi(0, G)$
- Or T_1 forces $\varphi(a, G)$ and $\neg\varphi(a+1, G)$ for some $a \in M$

Given $T \in \mathbb{P}$, define the $\Sigma^0_1(\mathcal{M})$ set

$$W = \{x \in M : T \,?\!\vdash \varphi(x, G)\}$$

- Case 1: $0 \notin W$.
  Then there is an extension forcing $\neg\varphi(0, G)$
- Case 2: $a \in W$ and $a + 1 \notin W$ for some $a \in M$
  Then there is an extension forcing $\varphi(a, G)$ and $\neg\varphi(a + 1, G)$
- Case 3: $0 \in W$ and $\forall a \in M$ $(a \in W \rightarrow a + 1 \in W)$
  Impossible, since $\mathcal{M} \models \mathsf{I}\Sigma^0_1$ but $b \notin W$.

Every set can be Δ^0_2
from the viewpoint of RCA_0.

Theorem (Towsner)

Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0$ be a countable model and $A \subseteq M$ be an arbitrary set. There is a set $G \subseteq M$ such that A is $\Delta^0_2(G)$ and $\mathcal{M}[G] \models \mathsf{RCA}_0$.

Towsner forcing

$\mathbb{P}$: set of pairs (g, I) in $\mathcal{M}$ such that

- $g \subseteq M^2 \to 2$ is a finite partial function;
- $I \subseteq M$ is a finite set of "locked" columns.

$[g, I]$: class of all partial functions $h \subseteq M^2 \to 2$ such that

- $g \subseteq h$;
- for all $(x, y) \in \operatorname{dom} h \setminus \operatorname{dom} g$, if $x \in I$ then $h(x, y) = A(x)$.

$(h, J) \leq (g, I)$ if $J \supseteq I$ and $h \in [g, I]$

$$(g, I) \;?\!\vdash\; \exists y \psi(y, G \restriction_y)$$

there is some $h \in [g, I]$ and some y such that $\psi(y, h \restriction_y)$.

Lemma

Let (g, I) be a condition and $\varphi(G)$ be a $\Sigma^0_1(\mathcal{M})$-formula.

1. If $(g, I) \;?\!\vdash \varphi(G)$ then there is an extension forcing $\varphi(G)$
2. If $(g, I) \;?\!\nvdash \varphi(G)$ then (g, I) forces $\neg\varphi(G)$

Lemma (Friedman)

Let $\mathcal{M} = (M, S) \models \mathsf{RCA}_0$ and $G \subseteq M$ be such that $\mathcal{M} \cup \{G\} \models \mathrm{I}\Sigma^0_1$. Then $\mathcal{M}[G] \models \mathsf{RCA}_0$.

Lemma

Let (g, I) be a condition and $\varphi(x, X)$ be a $\Sigma^0_1(\mathcal{M})$-formula such that (g, I) forces $\neg\varphi(b, G)$ for some $b \in M$. Then there is an extension $(h, J) \leq (g, I)$ such that

- Either (h, J) forces $\neg\varphi(0, G)$
- Or (h, J) forces $\varphi(a, G)$ and $\neg\varphi(a + 1, G)$ for some $a \in M$

Given $(g, I) \in \mathbb{P}$, define the $\Sigma^0_1(\mathcal{M})$ set

$$W = \{x \in M : (g, I) \mathrel{?\!\vdash} \varphi(x, G)\}$$

- Case 1: $0 \notin W$.
  Then there is an extension forcing $\neg\varphi(0, G)$
- Case 2: $a \in W$ and $a + 1 \notin W$ for some $a \in M$
  Then there is an extension forcing $\varphi(a, G)$ and $\neg\varphi(a + 1, G)$
- Case 3: $0 \in W$ and $\forall a \in M\ (a \in W \to a + 1 \in W)$
  Impossible, since $\mathcal{M} \models \mathrm{I}\Sigma^0_1$ but $b \notin W$.

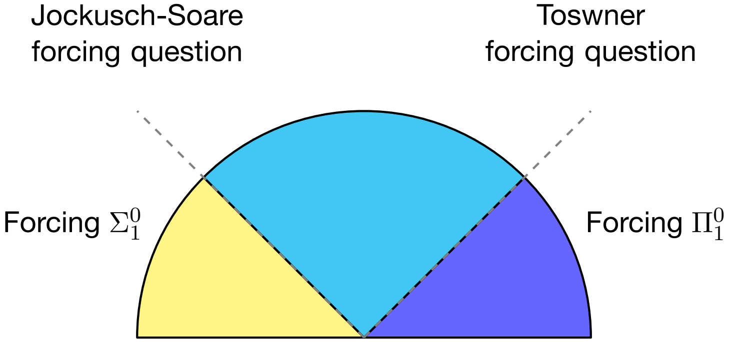
Fix a notion of forcing $(\mathbb{P}, \leq)$.

A forcing question is (Σ^0_n, Π^0_n)-merging if for every $p \in \mathbb{P}$ and every pair of Σ^0_n-formulas $\varphi(G), \psi(G)$ such that $p \mathrel{?\vdash} \varphi(G)$ and $p \mathrel{?\nvdash} \psi(G)$, there is an extension $q \leq p$ such that $q \Vdash \varphi(G) \wedge \neg\psi(G)$..

Lemma

Suppose $?\vdash$ is Σ^0_1-preserving and (Σ^0_1, Π^0_1)-merging. For every $\{0,1\}$-valued functional Φ_e, the following set is dense in $(\mathbb{P}, \leq)$.

$$D = \{p \in \mathbb{P} : p \Vdash \exists x\ \Phi^G_e(x) \uparrow \vee \exists x\ \Phi^G_e(x) \downarrow = \Phi_x(x)\}$$


Jockusch-Soare
forcing question
Toswner
forcing question
Forcing Σ^0_1
Forcing Π^0_1

What we know so far...

Forcing question $?\vdash$	Notion of forcing $(\mathbb{P}, \leq)$
Σ^0_1-preserving	cone avoidance
Σ^0_1-pres. and Σ^0_1-compact	pres. of hyperimmunity
Σ^0_1-pres. and Π^0_1-merging	PA avoidance
Σ^0_1-pres. and ω-Π^0_1-merging	DNC avoidance
Σ^0_1-pres. and (Σ^0_1, Π^0_1)-merging	IΣ^0_1 preservation
$\dots$	$\dots$

Higher jump control

An infinite set C is cohesive for a sequence $R_0, R_1, \dots$ if for every i, $C \subseteq^* R_i$ or $C \subseteq^* \overline{R_i}$

COH

Cohesiveness principle
Every sequence of sets admits a cohesive set

Cohesiveness is about
jump computation

Let $R_0, R_1, \ldots$ be an infinite sequence of sets

Given $\sigma \in 2^{<\mathbb{N}}$, let

$$\vec{R}_\sigma = \bigcap_{\sigma(i)=0} \overline{R}_i \bigcap_{\sigma(i)=1} R_i$$

Let $\mathcal{C}(\vec{R})$ be the $\Pi^0_1(\emptyset')$ class of all P such that for every $\sigma \prec P$, $\vec{R}_\sigma$ is infinite

Lemma

Let $\vec{R}$ be a uniformly computable sequence of sets.
A set computes an infinite $\vec{R}$-cohesive set iff its jump computes a member of $\mathcal{C}(\vec{R})$.

Lemma

For every $\Pi^0_1(\emptyset')$ class $\mathcal{P} \subseteq 2^{\mathbb{N}}$, there is a uniformly computable sequence of sets $\vec{R}$ such that $\mathcal{C}(\vec{R}) = \mathcal{P}$.

A function $f : \mathbb{N} \to \mathbb{N}$ is diagonally non-X-computable (X-DNC) if

$$\forall e\ f(e) \neq \Phi_e^X(e)$$

Lemma

There exists an X-computable infinite binary tree $T \subseteq 2^{<\mathbb{N}}$ such that $[T]$ are the $\{0,1\}$-valued X-DNC functions.

- $T = \{\sigma \in 2^{<\mathbb{N}} : \forall e < |\sigma|\ \sigma(e) \neq \Phi_e^X(e)[|\sigma|]\}$.

Lemma

For every X-computable infinite binary tree T, every $\{0,1\}$-valued X-DNC function computes a path.

- Given $\sigma \in T$ and $x \in \mathbb{N}$, let $\Phi_{e_\sigma}^X$ explore the branches below $\sigma \cdot 0$ and $\sigma \cdot 1$.
- If the branch below $\sigma \cdot i$ is the first to die, then halt and output i.
- For every σ extensible in T, $\sigma \cdot f(e_\sigma)$ is extensible in T.

Lemma

Let $\vec{R}$ be a uniformly computable sequence of sets.
Every set whose jump computes a $\{0,1\}$-valued $\emptyset'$-DNC function computes an infinite $\vec{R}$-cohesive set.

Lemma

There is a uniformly computable sequence of sets $\vec{R}$ such that for every $\vec{R}$-cohesive set, its jump computes a $\{0,1\}$-valued $\emptyset'$-DNC function.

Fix a notion of forcing $(\mathbb{P}, \leq)$.

A forcing question is Π^0_n-merging if for every $p \in \mathbb{P}$ and every pair of Σ^0_n-formulas $\varphi(G), \psi(G)$ such that $p \mathrel{?}\!\nvdash \varphi(G)$ and $p \mathrel{?}\!\nvdash \psi(G)$, there is an extension $q \leq p$ such that $q \Vdash \neg\varphi(G) \wedge \neg\psi(G)$.

Lemma

Suppose $?\vdash$ is Σ^0_n-preserving and Π^0_n-merging. For every $\{0,1\}$-valued functional Φ_e, the following set is dense in $(\mathbb{P}, \leq)$.

$$D = \{p \in \mathbb{P} : p \Vdash \exists x\, \Phi_e^{G^{(n-1)}}(x) \uparrow \vee \exists x\, \Phi_e^{G^{(n-1)}}(x) \downarrow = \Phi_x^{\emptyset^{(n-1)}}(x)\}$$

Given $p \in \mathbb{P}$, define the Σ^0_n set

$$W = \{(x, v) : p \mathrel{?\vdash} \Phi_e^{G^{(n-1)}}(x) \downarrow = v\}$$

- Case 1: $(x, \Phi_x^{\emptyset^{(n-1)}}(x)) \in W$ for some x such that $\Phi_x^{\emptyset^{(n-1)}}(x) \downarrow$
  Then τ is an extension forcing $\Phi_e^{G^{(n-1)}}(x) = \Phi_x^{\emptyset^{(n-1)}}(x)$
- Case 2: $(x, 0), (x, 1) \notin W$ for some x
  Then σ forces $\Phi_e^{G^{(n-1)}}(x) \uparrow$
- Case 3: W is a Σ^0_n graph of a $\emptyset^{(n-1)}$-DNC function
  Impossible, since no $\emptyset^{(n-1)}$-DNC function is $\emptyset^{(n-1)}$-computable.

Cohen forcing

$$(2^{<\omega}, \preceq)$$

$2^{<\omega}$ is the set of all finite binary strings

$\sigma \preceq \tau$ means σ is a prefix of τ

$[\sigma] = \{X \in 2^{\omega} : \sigma \prec X\}$

Theorem (Folklore)

Every sufficiently Cohen generic G computes no $\{0,1\}$-valued DNC function.

Lemma

For every $\{0,1\}$-valued Turing functional Φ_e, the following set is dense in $(2^{<\omega}, \preceq)$.

$$D = \{\sigma \in 2^{<\omega} : \sigma \Vdash \exists x\ \Phi_e^G(x) \uparrow \vee \exists x\ \Phi_e^G(x) \downarrow = \Phi_x(x)\}$$

Let $\sigma \in 2^{<\mathbb{N}}$ and $\varphi(G) \equiv \exists x \psi(G, x)$ be a Σ^0_n formula for $n \geq 1$.

$$\sigma \mathrel{?\!\!\vdash} \varphi(G) \equiv \begin{cases} \exists x\ \exists \tau \succeq \sigma\ \psi(\tau, x) & \text{for } n = 1 \\ \exists x\ \exists \tau \succeq \sigma\ \tau \mathrel{?\!\!\nvdash} \neg\psi(G, x) & \text{for } n > 1 \end{cases}$$

Lemma

The forcing question for Σ^0_n-formulas is Σ^0_n-preserving

Pigeonhole principle

RT^1_k Every k-partition of $\mathbb{N}$ admits an infinite subset of a part.

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	....

$\longmapsto$

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	....

Theorem (Dzhafarov and Jockusch)

For every set $C \not\leq_T \emptyset$ and every 2-partition $A_0 \sqcup A_1 = \mathbb{N}$, there is some $i < 2$ and an infinite set $G \subseteq A_i$ such that $C \not\leq_T G$.

Theorem (Monin and Patey)

For every set $C \not\leq_T \emptyset^{(n)}$ and every 2-partition $A_0 \sqcup A_1 = \mathbb{N}$, there is some $i < 2$ and an infinite set $G \subseteq A_i$ such that $C \not\leq_T G^{(n)}$.

$$(F_0, F_1, X)$$

Initial segment   Reservoir

- F_i is finite, X is infinite, $\max F_i < \min X$ (Mathias condition)
- $C \not\leq_T X$ (Weakness property)
- $F_i \subseteq A_i$ (Combinatorics)

Extension

$$(E_0, E_1, Y) \leq (F_0, F_1, X)$$

- $F_i \subseteq E_i$
- $Y \subseteq X$
- $E_i \setminus F_i \subseteq X$

Denotation

$$\langle G_0, G_1 \rangle \in [F_0, F_1, X]$$

- $F_i \subseteq G_i$
- $G_i \setminus F_i \subseteq X$

$$[E_0, E_1, Y] \subseteq [F_0, F_1, X]$$

Mathias condition

$$(F, X)$$

Initial segment Reservoir

F is finite, X is infinite,
$\max F < \min X$

Mathias extension

$$(E, Y) \leq (F, X)$$

$$F \subseteq E, Y \subseteq X, E \setminus F \subseteq X$$

Cylinder

$$[F, X] = \{G : F \subseteq G \subseteq F \cup X\}$$

A function $g : \mathbb{N} \to \mathbb{N}$ dominates $f : \mathbb{N} \to \mathbb{N}$ if $\forall^\infty x\, g(x) \geq f(x)$.

The principal function of an infinite set $X = \{x_0 < x_1 < \dots\}$ is the function $p_X : n \mapsto x_n$.

A Turing degree $\mathbf{d}$ is high if $\mathbf{d}' \geq \mathbf{0}''$.

Theorem (Martin domination)

A degree is high iff it computes a function dominating every computable function

Lemma

If G is sufficiently Mathias generic, then p_G dominates every computable function

- Let $f : \mathbb{N} \to \mathbb{N}$ be a total computable function and (F, X) be a Mathias condition
- Let $Y \subseteq X$ be such that $p_{F \cup Y}$ dominates f
- The extension (F, Y) forces p_G to dominate f

Mathias forcing produces sparse sets which computes fast-growing functions even when using computable reservoirs

Solution: restrict reservoirs

The only operations on the reservoirs are partitioning and trimming.

Definition

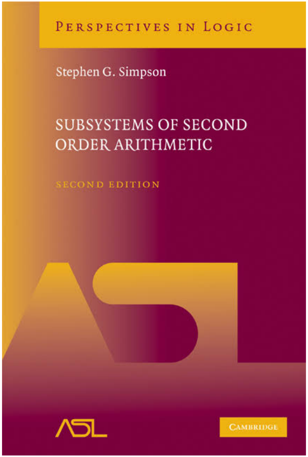
A non-empty class $\mathcal{P} \subseteq 2^{\mathbb{N}}$ is partition regular if

(1) For every $X \in \mathcal{P}$ and $Y \supseteq X$, $Y \in \mathcal{P}$

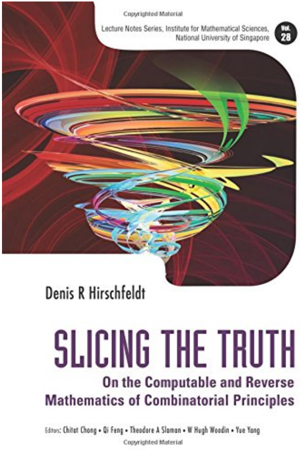
(2) For every $X \in \mathcal{P}$ and every $Z_0 \cup Z_1 = X$, there is some $i < 2$ such that $Z_i \in \mathcal{P}$

Conclusion

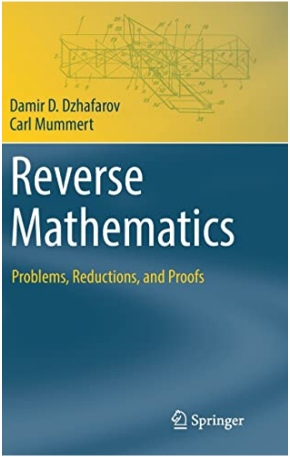
The computability-theoretic properties of forcing notions are consequences of combinatorial and definitional features of their forcing questions.



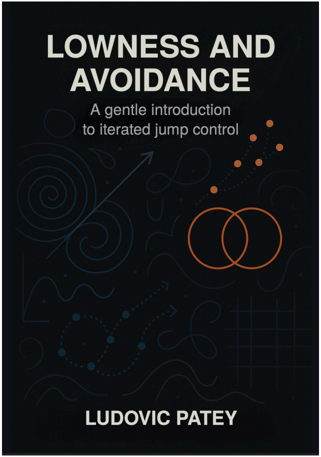
Subsystems of second-order arithmetic, 2010



Slicing the truth, 2014



Reverse Mathematics, 2022



Lowness and avoidance, 2025