Ramsey's theorem under a computable perspective

Ludovic PATEY

What is Ramsey's theorem?

RAMSEY'S THEOREM

$[X]^n$ is the set of unordered n-tuples of elements of X

A k-coloring of $[X]^n$ is a map $f : [X]^n \to k$

A set $H \subseteq X$ is homogeneous for f if $|f([X]^n)| = 1$.

RT^n_k Every k-coloring of $[\mathbb{N}]^n$ admits an infinite homogeneous set.

PIGEONHOLE PRINCIPLE

RT^1_k Every k-partition of $\mathbb{N}$ admits an infinite part.

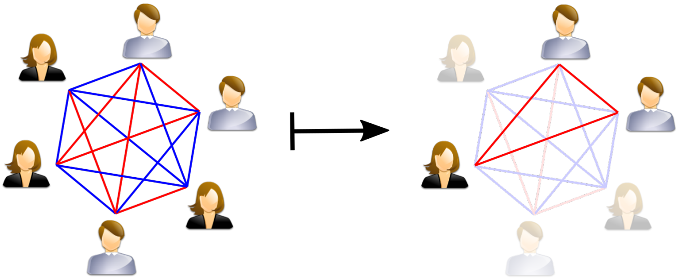
0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	....

$\longmapsto$

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	....

RAMSEY'S THEOREM FOR PAIRS

RT^2_k Every k-coloring of the infinite clique admits an infinite monochromatic subclique.



Why do we care about Ramsey's theorem?

REVERSE MATHEMATICS

Foundational program that seeks to determine the optimal axioms of ordinary mathematics.

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$$\mathrm{RCA}_0 \vdash A \leftrightarrow T$$

in a very weak theory RCA_0
capturing computable mathematics

RCA_0

Robinson arithmetics

$$m+1 \neq 0$$
$$m+1 = n+1 \rightarrow m = n$$
$$\neg(m < 0)$$
$$m < n+1 \leftrightarrow (m < n \vee m = n)$$

$$m+0 = m$$
$$m+(n+1) = (m+n)+1$$
$$m \times 0 = 0$$
$$m \times (n+1) = (m \times n)+m$$

Σ^0_1 induction scheme

$$\varphi(0) \wedge \forall n(\varphi(n) \Rightarrow \varphi(n+1)) \Rightarrow \forall n \varphi(n)$$

where $\varphi(n)$ is Σ^0_1

Δ^0_1 comprehension scheme

$$\forall n(\varphi(n) \Leftrightarrow \psi(n)) \Rightarrow \exists X \forall n(n \in X \Leftrightarrow \varphi(n))$$

where $\varphi(n)$ is Σ^0_1 with free X, and ψ is Π^0_1.

Reverse mathematics

Mathematics are computationally very structured

Almost every theorem is empirically equivalent to one among five big subsystems.

$$\Pi^1_1\mathsf{CA} \to \mathsf{ATR} \to \mathsf{ACA} \to \mathsf{WKL} \to \mathsf{RCA}_0$$

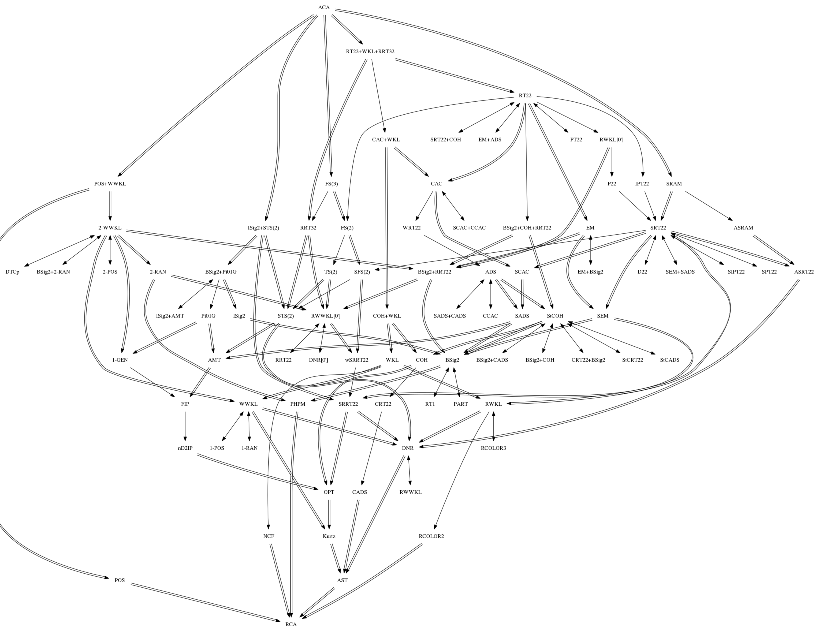
REVERSE MATHEMATICS

Mathematics are computationally very structured

Almost every theorem is empirically equivalent to one among five big subsystems.

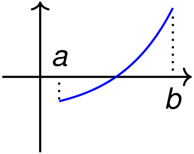
Except for Ramsey's theory...

$$\begin{tikzcd} & \Pi^1_1\mathrm{CA} \arrow[d] \\ & \mathrm{ATR} \arrow[d] \\ & \mathrm{ACA} \arrow[d] \arrow[dl] \\ \mathrm{RT}^2_2 \arrow[dr] & \mathrm{WKL} \arrow[d] \\ & \mathrm{RCA}_0 \end{tikzcd}$$

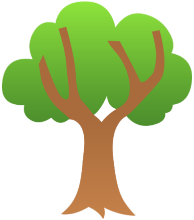

ACA
RT22+WKL+RRT32
RT22
CAC+WKL
SRT22+COH
EM+ADS
PT22
RWKL[0']
POS+WWKL
FS(3)
CAC
P22
IPT22
SRAM
2-WWKL
ISig2+STS(2)
RRT32
FS(2)
WRT22
SCAC+CCAC
BSig2+COH+RRT22
EM
SRT22
ASRAM
DTCp
BSig2+2-RAN
2-POS
2-RAN
BSig2+Pi01G
TS(2)
SFS(2)
BSig2+RRT22
ADS
SCAC
EM+BSig2
D22
SEM+SADS
SIPT22
SPT22
ASRT22
ISig2+AMT
Pi01G
ISig2
STS(2)
RWWKL[0']
COH+WKL
SADS+CADS
CCAC
SADS
StCOH
SEM
1-GEN
AMT
RRT22
DNR[0']
wSRRT22
WKL
COH
BSig2
BSig2+CADS
BSig2+COH
CRT22+BSig2
StCRT22
StCADS
FIP
WWKL
PHPM
SRRT22
CRT22
RT1
PART
RWKL
nD2IP
1-POS
1-RAN
DNR
RCOLOR3
OPT
CADS
RWWKL
NCF
Kurtz
RCOLOR2
POS
AST
RCA

Many theorems can be seen as problems.

Intermediate value theorem
For every continuous function f over an interval $[a, b]$ such that $f(a) \cdot f(b) < 0$, there is a real $x \in [a, b]$ such that $f(x) = 0$.



König's lemma
Every infinite, finitely branching tree admits an infinite path.



A problem P has a collection of instances $\mathcal{I}(\mathsf{P})$.
Every instance I has set of solutions $\mathcal{S}(I)$.

$\mathcal{I}(\mathsf{RT}^n_k) = \{f : [\mathbb{N}]^n \to k\}$

$\mathcal{S}(f) = \{$ infinite f-homogeneous set $\}$

$[X]^\omega$ denotes the set of infinite subsets of X

A problem P is of Ramsey-type if for every instance I, the set of solutions is dense and closed downward in $([\mathbb{N}]^\omega, \subseteq)$:

$$\forall X \in [\mathbb{N}]^\omega,\ [X]^\omega \cap \mathcal{S}(I) \neq \emptyset$$

$$\forall X \in \mathcal{S}(I),\ [X]^\omega \subseteq \mathcal{S}(I)$$

We can solve Ramsey-type problems simultaneously.

Given two Ramsey-type problems P and Q, define the problem

$$\mathsf{P} \cap \mathsf{Q} = \begin{cases} \mathcal{I}(\mathsf{P} \cap \mathsf{Q}) = \mathcal{I}(\mathsf{P}) \times \mathcal{I}(\mathsf{Q}) \\ \mathcal{S}(I, J) = \mathcal{S}(I) \cap \mathcal{S}(J) \end{cases}$$

What sets can encode Ramsey's theorem?

Fix a problem P.

A set S is P-encodable if there is an instance of P such that every solution computes S.

What sets can encode an instance of RT^n_k?

A function f is a modulus of a set S if every function dominating f computes S.

A set S is computably encodable if for every infinite set X, there is an infinite subset $Y \subseteq X$ computing S.

Thm (Solovay, Groszek and Slaman)

Given a set S, TFAE

- S is computably encodable
- S admits a modulus
- S is hyperarithmetic

Thm (Jockusch)

A set is RT^n_k-encodable for some $n \geq 2$ iff it is hyperarithmetic.

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A set is RT^n_k-encodable for some $n \geq 2$ iff it is hyperarithmetic.

Proof ($\Rightarrow$).

Let $g : [\omega]^n \to k$ be a coloring whose homogeneous sets compute S.

Since every infinite set has a homogeneous subset, S is computably encodable.

Thus S is hyperarithmetic. $\square$

Thm (Jockusch)

A set is RT^n_k-encodable for some $n \geq 2$ iff it is hyperarithmetic.

Proof ($\Leftarrow$).

Let S be hyperarithmetic with modulus μ_S.

Define $g : [\omega]^2 \to 2$ by $g(x, y) = 1$ iff $y > \mu_S(x)$.

Let $H = \{x_0 < x_1 < \dots\}$ be an infinite g-homogeneous set.

The function $p_H(n) = x_n$ dominates μ_S, hence computes S. $\square$

The encodability power of RT^n_k comes from the

sparsity

of its homogeneous sets.

What about RT^1_k?

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
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Sparsity of red implies non-sparsity of blue and conversely.

Thm (Dzhafarov and Jockusch)

A set is RT^1_2-encodable iff it is computable.

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A set is RT^1_2-encodable iff it is computable.

Input : a set $S \not\leq_T \emptyset$ and a 2-partition $A_0 \sqcup A_1 = \mathbb{N}$

Output : an infinite set $G \subseteq A_i$ such that $S \not\leq_T G$

$$(F_0, F_1, X)$$

Initial segment

Reservoir

- F_i is finite, X is infinite, $\max F_i < \min X$ (Mathias condition)
- $S \not\leq_T X$ (Weakness property)
- $F_i \subseteq A_i$ (Combinatorics)

Extension

$(E_0, E_1, Y) \leq (F_0, F_1, X)$

- $F_i \subseteq E_i$
- $Y \subseteq X$
- $E_i \setminus F_i \subseteq X$

Satisfaction

$\langle G_0, G_1 \rangle \in [F_0, F_1, X]$

- $F_i \subseteq G_i$
- $G_i \setminus F_i \subseteq X$

$$[E_0, E_1, Y] \subseteq [F_0, F_1, X]$$

$$(F_0, F_1, X) \Vdash \varphi(G_0, G_1)$$

Condition

Formula

$\varphi(G_0, G_1)$ holds for every $\langle G_0, G_1 \rangle \in [F_0, F_1, X]$

Input : a set $S \not\leq_T \emptyset$ and a 2-partition $A_0 \sqcup A_1 = \mathbb{N}$

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$$\Phi_{e_0}^{G_0} \neq S \vee \Phi_{e_1}^{G_1} \neq S$$

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$$\Phi_{e_0}^{G_0} \neq S \vee \Phi_{e_1}^{G_1} \neq S$$

The set $\left\{ \begin{array}{rl} c : c \Vdash (\exists x) & \Phi_{e_0}^{G_0}(x) \downarrow \neq S(x) \vee \Phi_{e_0}^{G_0}(x) \uparrow \\ \vee & \Phi_{e_1}^{G_1}(x) \downarrow \neq S(x) \vee \Phi_{e_1}^{G_1}(x) \uparrow \end{array} \right\}$ is dense

FIRST ATTEMPT

Given a condition $c = (F_0, F_1, X)$, suppose the formula

$$\varphi(x, n) = (\exists d \leq c) d \Vdash \Phi_{e_0}^{G_0}(x) \downarrow = n$$

is $\Sigma_1^{0,X}$ (it is not). Then the set

$$\mathcal{C} = \{(x, n) : \varphi(x, n)\}$$

is X-c.e.

FIRST ATTEMPT

$$\mathcal{C} = \{(x, n) : \varphi(x, n)\}$$

Σ_1 case

$(\exists x)(x, 1 - S(x)) \in \mathcal{C}$

Then $\exists d \leq c$ such that

$d \Vdash \Phi_{e_0}^{G_0}(x) \downarrow = 1 - S(x)$

Π_1 case

$(\exists x)(x, S(x)) \notin \mathcal{C}$

Then

$c \Vdash \Phi_{e_0}^{G_0}(x) \neq S(x)$

Impossible case

$(\forall x)(x, 1 - S(x)) \notin \mathcal{C}$

$(\forall x)(x, S(x)) \in \mathcal{C}$

Then since $\mathcal{C}$ is X-c.e

$S \leq_T X$ ↯

THE FIRST ATTEMPT FAILS

Given a condition $c = (F_0, F_1, X)$, the formula

$$\varphi(x, n) = (\exists d \leq c) d \Vdash \Phi_{e_0}^{G_0}(x) \downarrow = n$$

is too complex because it can be translated in

$$(\exists E_0 \subseteq X \cap A_0) \Phi_{e_0}^{F_0 \cup E_0}(x) \downarrow = n$$

which is $\Sigma_1^{0,A \oplus X}$ and not $\Sigma_1^{0,X}$.

IDEA: MAKE AN OVERAPPROXIMATION

"Can we find an extension for every instance of RT^1_2?"

Given a condition $c = (F_0, F_1, X)$, let $\psi(x, n)$ be the formula

$$(\forall B_0 \sqcup B_1 = \mathbb{N})(\exists i < 2)(\exists E_i \subseteq X \cap B_i)\Phi_{e_i}^{F_i \cup E_i}(x) \downarrow = n$$

$$\psi(x, n) \text{ is } \Sigma_1^{0,X}$$

Case 1: $\psi(x, n)$ holds

Letting $B_i = A_i$, there is an extension $d \leq c$ forcing

$$\Phi_{e_0}^{G_0}(x) \downarrow= n \vee \Phi_{e_1}^{G_1}(x) \downarrow= n$$

Case 2: $\psi(x, n)$ does not hold

$$(\exists B_0 \sqcup B_1 = \mathbb{N})(\forall i < 2)(\forall E_i \subseteq X \cap B_i)\Phi_{e_i}^{F_i \cup E_i}(x) \neq n$$

The condition $(F_0, F_1, X \cap B_i) \leq c$ forces

$$\Phi_{e_0}^{G_0}(x) \neq n \vee \Phi_{e_1}^{G_1}(x) \neq n$$

SECOND ATTEMPT

$$\mathcal{D} = \{(x, n) : \psi(x, n)\}$$

Σ_1 case

$(\exists x)(x, 1 - S(x)) \in \mathcal{D}$

Then $\exists d \leq c\ \exists i < 2$

$d \Vdash \Phi_{e_i}^{G_i}(x) \downarrow = 1 - S(x)$

Π_1 case

$(\exists x)(x, S(x)) \notin \mathcal{D}$

Then $\exists d \leq c\ \exists i < 2$

$d \Vdash \Phi_{e_i}^{G_i}(x) \neq S(x)$

Impossible case

$(\forall x)(x, 1 - S(x)) \notin \mathcal{D}$

$(\forall x)(x, S(x)) \in \mathcal{D}$

Then since $\mathcal{D}$ is X-c.e

$S \leq_T X$ ↯

CJS ARGUMENT

Context: We build a solution G to a P-instance X

Goal: Decide a property $\varphi(G)$.

Question: For every P-instance Y, can I find a solution G satisfying $\varphi(G)$?

If yes: In particular for $Y = X$, I can satisfy $\varphi(G)$.

If no: If no: By making G be a solution to X and Y simultaneously, I will satisfy $\neg\varphi(G)$.

Mathias forcing

with a

CJS argument

are sufficient to compare Ramsey-type statements.

RAMSEY'S THEOREM

Over n-tuples

$$\mathrm{RT}^{n}_{k}$$

Using k colors

RAMSEY'S THEOREM

Over n-tuples

$$\mathrm{RT}^{n}_{k,r}$$

Allows r colors

Using k colors

Thm (Wang)

A set is $\mathsf{RT}^n_{k,\ell}$-encodable iff it is computable for large ℓ

(whenever ℓ is at least the nth Schröder Number)

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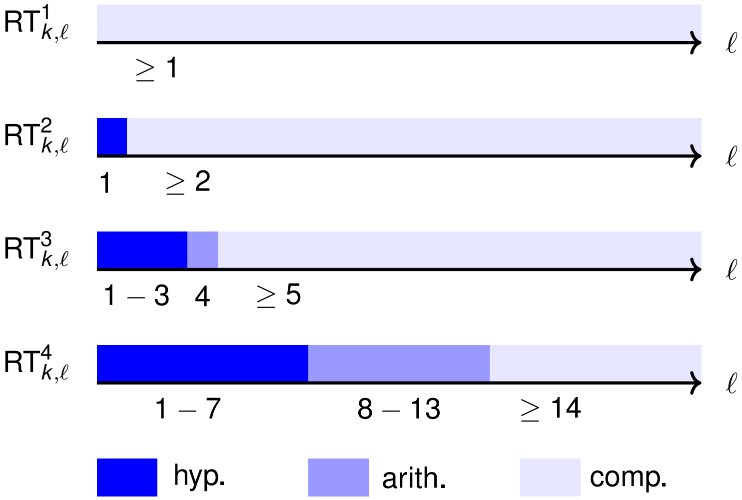
Thm (Cholak, P.)

A set is $\mathsf{RT}^n_{k,\ell}$-encodable iff it is arithmetic for medium ℓ

$\mathsf{RT}^n_{k,\ell}$-ENCODABLE SETS



How simple can be the solutions of a computable instance of RT^n_k?

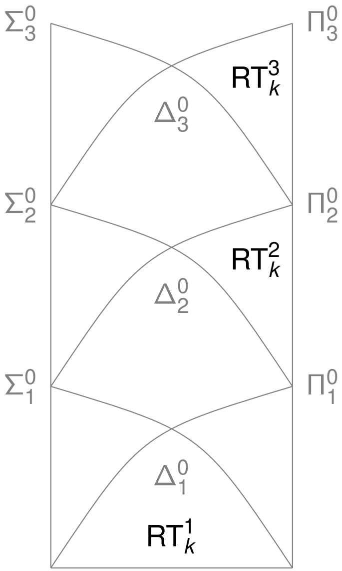
Fix some $n \geq 2$.

Thm (Jockusch)

Every computable instance of RT^n_k has a Π^0_n solution.

Thm (Jockusch)

There is a computable instance of RT^n_k with no Σ^0_n solution.



$f : [\mathbb{N}]^{n+1} \to k$ is stable if for every $\sigma \in [\mathbb{N}]^n$, $\lim_y f(\sigma, y)$ exists.

If $f : [\mathbb{N}]^{n+1} \to k$ is stable, define $\tilde{f}(\sigma) = \lim_y f(\sigma, y)$.

Given a problem P, define its jump

$$\mathsf{J}(\mathsf{P}) = \begin{cases} \mathcal{I}(\mathsf{J}(\mathsf{P})) = \{f : \tilde{f} \in \mathcal{I}(\mathsf{P})\} \\ \mathcal{S}_{\mathsf{J}(\mathsf{P})}(f) = \mathcal{S}_{\mathsf{P}}(\tilde{f}) \end{cases}$$

SRT^n_k : RT^n_k restricted to stable colorings.

$\emptyset'$-computable | stable computable

$$\mathsf{RT}^n_k \quad \Leftrightarrow \quad \mathsf{RT}^{n+1}_k$$

SRT^n_k : RT^n_k restricted to stable colorings.

$\emptyset'$-computable		stable computable
RT^n_k	$\Leftrightarrow$	RT^{n+1}_k

"Every Δ^0_2 set has an infinite subset or cosubset" $\Leftrightarrow$ SRT^2_2

A set S is computably P-encodable if there is a computable instance of P whose solutions compute S.

Thm

$\emptyset'$ is computably $\mathsf{J}(\mathsf{RT}^2_2)$-encodable.

$$f(x,y) = \begin{cases} 1 & \text{if } y \geq \mu_{\emptyset'}(x) \\ 0 & \text{otherwise} \end{cases}$$

Cor (Jockusch)

$\emptyset^{(n)}$ is computably RT^{n+2}_2-encodable.

Given a problem P, define its finite-error version

$$\mathsf{P}^* = \begin{cases} \mathcal{I}(\mathsf{P}^*) = \mathcal{I}(\mathsf{P}) \\ \mathcal{S}_{\mathsf{P}^*}(I) = \{Y : \exists Z \in \mathcal{S}_{\mathsf{P}}(I),\ Z =^* Y\} \end{cases}$$

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An infinite set C is $\vec{R}$-cohesive for some sets $R_0, R_1, \ldots$ if for every i, either $C \subseteq^* R_i$ or $C \subseteq^* \overline{R}_i$.

$\bigcap \mathsf{RT}^{1*}_2$: Every collection of sets has a cohesive set.

RT^{n+1}_k follows from $\bigcap \mathsf{RT}^{1*}_2$ and $\mathsf{J}(\mathsf{RT}^n_k)$.

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Given $f : [\mathbb{N}]^{n+1} \to k$, define $\langle R_{\sigma,c} : \sigma \in [\mathbb{N}]^n, c < k \rangle$ by

$$R_{\sigma,c} = \{y : f(\sigma, y) = c\}$$

By $\bigcap \mathrm{RT}^{1*}_2$, there is an $\vec{R}$-cohesive set C.

RT^{n+1}_k follows from $\bigcap \mathsf{RT}^{1*}_2$ and $\mathsf{J}(\mathsf{RT}^n_k)$.

Given $f : [\mathbb{N}]^{n+1} \to k$, define $\langle R_{\sigma,c} : \sigma \in [\mathbb{N}]^n, c < k \rangle$ by

$$R_{\sigma,c} = \{y : f(\sigma, y) = c\}$$

By $\bigcap \mathsf{RT}^{1*}_2$, there is an $\vec{R}$-cohesive set C.

$f[C]^{n+1} \to k$ is an instance of $\mathsf{J}(\mathsf{RT}^n_k)$

By $\mathsf{J}(\mathsf{RT}^n_k)$, there is an infinite $\tilde{f}$-homogeneous set H

$H \oplus C$ computes an infinite f-homogeneous set

A set S is (computably) P-encodable if there is a (computable) instance of P whose solutions compute S.

Thm (Dzhafarov, Jockusch; Wang)

The RT^1_k-encodable and the $\bigcap \mathsf{RT}^{1*}_2$-encodable sets are the computable ones.

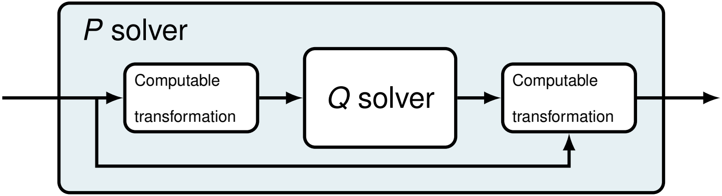
Cor (Seetapun)

The computably RT^2_k-encodable sets are the computable ones.

The combinatorial features of RT^n_k reveal the computational features of RT^{n+1}_k

How do Ramsey-type problems compare?

COMPUTABLE REDUCTION



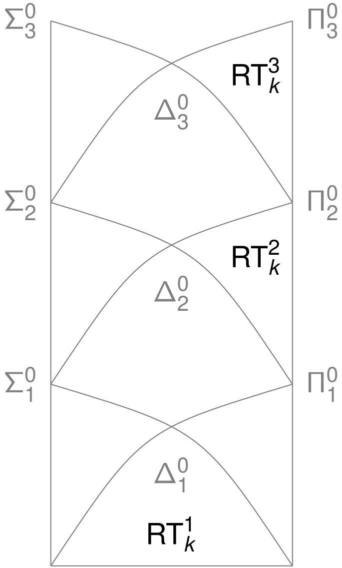
$$P \leq_c Q$$

Every P-instance I computes a Q-instance J such that for every solution X to J, $X \oplus I$ computes a solution to I.

Thm (Jockusch)

For every $n \geq 1$, $\mathrm{RT}^{n+1}_k \not\leq_c \mathrm{RT}^n_k$.

Proof: RT^n_k has Π^0_n solutions, but RT^{n+1}_k doesn't.



A function f is hyperimmune if it is not dominated by a computable function.

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A problem P preserves ℓ among k hyperimmunities if for every k-tuple $f_1, \dots, f_k$ of hyperimmune functions and every computable P-instance I, there is a solution Y such that at least ℓ among k of the f_i are Y-hyperimmune.

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A problem P preserves ℓ among k hyperimmunities if for every k-tuple $f_1, \dots, f_k$ of hyperimmune functions and every computable P-instance I, there is a solution Y such that at least ℓ among k of the f_i are Y-hyperimmune.

Thm (P.)

RT^2_k preserves 2 among $k+1$ hyperimmunities, but not RT^2_{k+1}.

Cor (P.)

$\mathsf{RT}^2_{k+1} \not\leq_c \mathsf{RT}^2_k$.

What is left?

Have we found the right framework?

Can Mathias forcing and the CJS argument answer all the Ramsey-type questions?

The CJS argument applied to RT^1_2 yields solutions to $\bigcap \mathsf{RT}^{1*}_2$.

Fix a computable sequence of sets $R_0, R_1, \ldots$

Is there a set X, such that every infinite set $H \subseteq X$ or $H \subseteq \overline{X}$ computes an $\vec{R}$-cohesive set?

A set X is high if $X' \geq_T \emptyset''$.

Is there a set X, such that every infinite set $H \subseteq X$ or $H \subseteq \overline{X}$ is high?

If yes, then $\bigcap \mathsf{RT}^{1*}_2 \leq_{oc} \mathsf{RT}^1_2$.

If no, well, this is still interesting *per se*.

A set S is P-jump-encodable if there is an instance of P such that the jump of every solution computes S.

Are the RT^1_2-jump-encodable sets precisely the $\emptyset'$-computable ones?

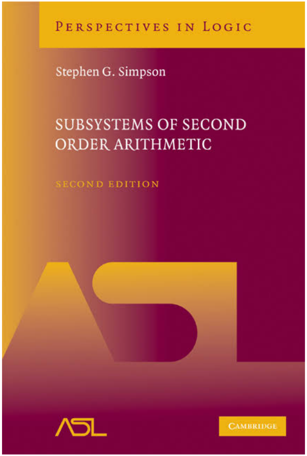
CONCLUSION

We have a minimalistic framework which answers accurately many questions about Ramsey's theorem.

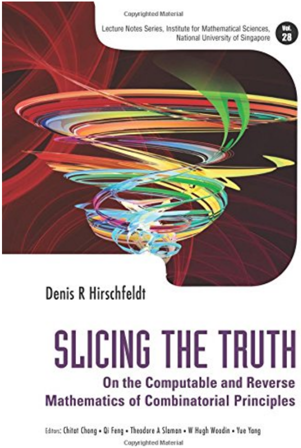
Ramsey-type problems compute through sparsity.

The computational properties of Ramsey-type problems are often immediate consequences of their combinatorics.

We understand what the Ramsey-type problems compute, but ignore what the jump of their solutions compute.



Subsystems of second-order arithmetic



Slicing the truth

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