Can we fish with Mathias forcing?

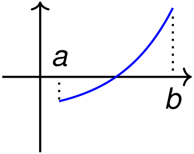
Ludovic PATEY

September 8, 2017

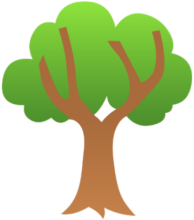
Introduction

Many theorems can be seen as problems.

Intermediate value theorem
For every continuous function f over an interval $[a, b]$ such that $f(a) \cdot f(b) < 0$, there is a real $x \in [a, b]$ such that $f(x) = 0$.



König's lemma
Every infinite, finitely branching tree admits an infinite path.



REVERSE MATHEMATICS

Foundational program that seeks to determine the optimal axioms of ordinary mathematics.

REVERSE MATHEMATICS

Foundational program that seeks to determine the optimal axioms of ordinary mathematics.

$$\mathsf{RCA}_0 \vdash A \leftrightarrow T$$

in a very weak theory RCA_0
capturing computable mathematics

RCA_0

Robinson arithmetics

$$m + 1 \neq 0$$
$$m + 1 = n + 1 \rightarrow m = n$$
$$\neg(m < 0)$$
$$m < n + 1 \leftrightarrow (m < n \lor m = n)$$

$$m + 0 = m$$
$$m + (n + 1) = (m + n) + 1$$
$$m \times 0 = 0$$
$$m \times (n + 1) = (m \times n) + m$$

Σ^0_1 induction scheme

$$\varphi(0) \land \forall n(\varphi(n) \Rightarrow \varphi(n+1)) \Rightarrow \forall n \varphi(n)$$

where $\varphi(n)$ is Σ^0_1

Δ^0_1 comprehension scheme

$$\forall n(\varphi(n) \Leftrightarrow \psi(n)) \Rightarrow \exists X \forall n(n \in X \Leftrightarrow \varphi(n))$$

where $\varphi(n)$ is Σ^0_1 with free X, and ψ is Π^0_1.

REVERSE MATHEMATICS

Mathematics are computationally very structured

Almost every theorem is empirically equivalent to one among five big subsystems.

$$\Pi^1_1\mathrm{CA} \to \mathrm{ATR} \to \mathrm{ACA} \to \mathrm{WKL} \to \mathrm{RCA}_0$$

REVERSE MATHEMATICS

Mathematics are computationally very structured

Almost every theorem is empirically equivalent to one among five big subsystems.

Except for Ramsey's theory...

$$\begin{tikzcd}
 & \Pi^1_1\mathrm{CA} \arrow[d] \\
 & \mathrm{ATR} \arrow[d] \\
 & \mathrm{ACA} \arrow[d] \arrow[dl] \\
\mathrm{RT}^2_2 \arrow[dr] & \mathrm{WKL} \arrow[d] \\
 & \mathrm{RCA}_0
\end{tikzcd}$$

RAMSEY'S THEOREM

$[X]^n$ is the set of unordered n-tuples of elements of X

A k-coloring of $[X]^n$ is a map $f : [X]^n \to k$

A set $H \subseteq X$ is homogeneous for f if $|f([H]^n)| = 1$.

RT^n_k Every k-coloring of $[\mathbb{N}]^n$ admits an infinite homogeneous set.

PIGEONHOLE PRINCIPLE

RT^1_k Every k-partition of $\mathbb{N}$ admits an infinite part.

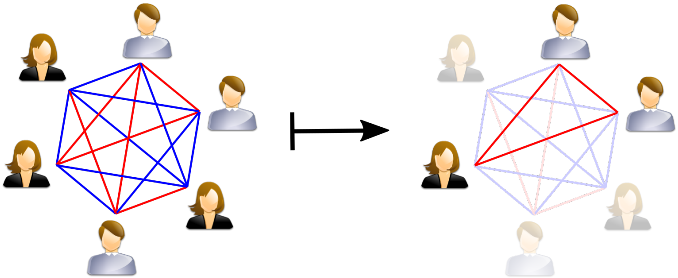
0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	....

$\longmapsto$

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	....

RAMSEY'S THEOREM FOR PAIRS

RT^2_k Every k-coloring of the infinite clique admits an infinite monochromatic subclique.



$$\mathsf{RCA}_0 \nvdash \mathsf{RT}^2_2 \to \mathsf{ACA}$$

(Seetapun)

By preserving a weakness property
using a proto version of the CJS argument.

A weakness property is a collection of sets closed downwards under the Turing reduction.

Examples

- $\{X : X \text{ is low}\}$
- $\{X : A \not\leq_T X\}$ for some set A
- $\{X : X \text{ is hyperimmune-free}\}$

Fix a weakness property $\mathcal{W}$.

A problem P preserves $\mathcal{W}$ if for every $Z \in \mathcal{W}$, every Z-computable P-instance X has a solution Y such that $Y \oplus Z \in \mathcal{W}$

Lemma

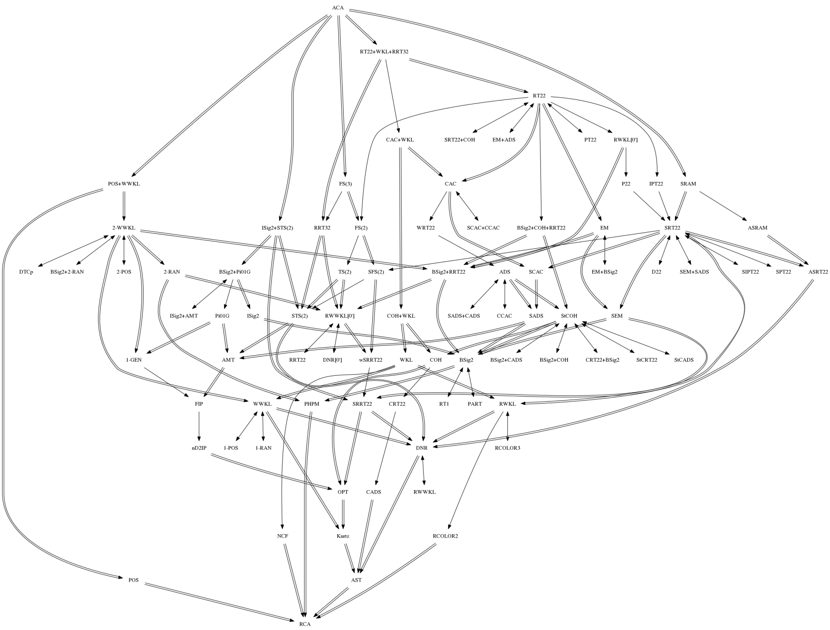
If P *preserves* $\mathcal{W}$ *but* Q *does not, then* $\mathsf{RCA}_0 \nvdash \mathsf{P} \to \mathsf{Q}$

$$\mathsf{RCA}_0 \nvdash \mathsf{RT}^2_2 \rightarrow \mathsf{ACA}$$

(Seetapun)

By preserving $\mathcal{W} = \{X : X \text{ is incomplete }\}$ using a proto version of the CJS argument.

The success of Mathias forcing and the CJS argument



ACA
RT22+WKL+RRT32
RT22
CAC+WKL
SRT22+COH
EM+ADS
PT22
RWKL[0']
POS+WWKL
FS(3)
CAC
P22
IPT22
SRAM
2-WWKL
ISig2+STS(2)
RRT32
FS(2)
WRT22
SCAC+CCAC
BSig2+COH+RRT22
EM
SRT22
ASRAM
DTCp
BSig2+2-RAN
2-POS
2-RAN
BSig2+Pi01G
TS(2)
SFS(2)
BSig2+RRT22
ADS
SCAC
EM+BSig2
D22
SEM+SADS
SIPT22
SPT22
ASRT22
ISig2+AMT
Pi01G
ISig2
STS(2)
RWWKL[0']
COH+WKL
SADS+CADS
CCAC
SADS
StCOH
SEM
1-GEN
AMT
RRT22
DNR[0']
wSRRT22
WKL
COH
BSig2
BSig2+CADS
BSig2+COH
CRT22+BSig2
StCRT22
StCADS
FIP
WWKL
PHPM
SRRT22
CRT22
RT1
PART
RWKL
nD2IP
1-POS
1-RAN
DNR
RCOLOR3
OPT
CADS
RWWKL
NCF
Kurtz
RCOLOR2
POS
AST
RCA

Separations are often achieved by preserving weakness properties using **canonical notions of forcing**

Separations by weakness properties

- $\mathsf{WKL} \nvdash_c \mathsf{ACA}$ (cone avoidance)
- $\mathsf{RT}^2_2 \nvdash_c \mathsf{ACA}$ (cone avoidance)
- $\mathsf{EM} \nvdash_c \mathsf{RT}^2_2$ (2 hyperimmunities)
- $\mathsf{EM} \nvdash_c \mathsf{TS}^2$ (ω hyperimmunities)
- $\mathsf{TS}^2 \nvdash_c \mathsf{RT}^2_2$ (2 hyperimmunities)
- $\mathsf{RT}^2_2 \nvdash_c \mathsf{TT}^2_2$ (fairness property)
- $\mathsf{RT}^2_2 \nvdash_c \mathsf{WWKL}$ (c.b-enum avoidance)
- ...

A notion of forcing $\mathbb{P}$ is canonical for a problem P if the properties preserved by the problem and by the notion of forcing coincide.

Restriction to classes of properties

FAMILIES OF PROPERTIES

Effectiveness

- Lowness
- Hyperimmune-freeness
- Hyperarithmetic
- ...

Genericity

- Cone avoidance
- Preservation of non-Σ^0_n definitions
- Preservation of hyperimmunity
- ...

EXAMPLE

$\mathcal{P}$ is an open genericity property if $\mathcal{P}$ is the set of oracles which do not compute a member of a fixed closed set $\mathcal{C} \subseteq \omega^\omega$

Contains already all the genericity properties used in reverse mathematics.

Theorem (Hirschfeldt and P.)

WKL *and the notion of forcing with* Π^0_1 *classes preserve the same open genericity properties*

Mathias forcing

with a

CJS argument

are sufficient to analyse
Ramsey-type statements.

$[X]^\omega$ denotes the set of infinite subsets of X

A problem P is of Ramsey-type if for every instance I, the set of solutions is dense and closed downward in $([\mathbb{N}]^\omega, \subseteq)$:

$$\forall X \in [\mathbb{N}]^\omega,\ [X]^\omega \cap \mathcal{S}(I) \neq \emptyset$$

$$\forall X \in \mathcal{S}(I),\ [X]^\omega \subseteq \mathcal{S}(I)$$

We can solve Ramsey-type problems simultaneously.

Given two Ramsey-type problems P and Q, define the problem

$$\mathsf{P} \cap \mathsf{Q} = \begin{cases} \mathcal{I}(\mathsf{P} \cap \mathsf{Q}) = \mathcal{I}(\mathsf{P}) \times \mathcal{I}(\mathsf{Q}) \\ \mathcal{S}(I, J) = \mathcal{S}(I) \cap \mathcal{S}(J) \end{cases}$$

Thm (Dzhafarov and Jockusch)

If a set S is not computable, then for every set A, there is an infinite set $G \subseteq A$ or $G \subseteq \overline{A}$ such that $S \not\leq_T G$.

Thm (Dzhafarov and Jockusch)

If a set S is not computable, then for every set A, there is an infinite set $G \subseteq A$ or $G \subseteq \overline{A}$ such that $S \not\leq_T G$.

Input : a set $S \not\leq_T \emptyset$ and a 2-partition $A_0 \sqcup A_1 = \mathbb{N}$

Output : an infinite set $G \subseteq A_i$ such that $S \not\leq_T G$

$$(F_0, F_1, X)$$

Initial segment Reservoir

- F_i is finite, X is infinite, $\max F_i < \min X$ (Mathias condition)
- $S \not\leq_T X$ (Weakness property)
- $F_i \subseteq A_i$ (Combinatorics)

Extension

$(E_0, E_1, Y) \leq (F_0, F_1, X)$

- $F_i \subseteq E_i$
- $Y \subseteq X$
- $E_i \setminus F_i \subseteq X$

Satisfaction

$\langle G_0, G_1 \rangle \in [F_0, F_1, X]$

- $F_i \subseteq G_i$
- $G_i \setminus F_i \subseteq X$

$$[E_0, E_1, Y] \subseteq [F_0, F_1, X]$$

$$(F_0, F_1, X) \Vdash \varphi(G_0, G_1)$$

Condition Formula

$\varphi(G_0, G_1)$ holds for every $\langle G_0, G_1 \rangle \in [F_0, F_1, X]$

Input : a set $S \not\leq_T \emptyset$ and a 2-partition $A_0 \sqcup A_1 = \mathbb{N}$

Output : an infinite set $G \subseteq A_i$ such that $S \not\leq_T G$

Input : a set $S \not\leq_T \emptyset$ and a 2-partition $A_0 \sqcup A_1 = \mathbb{N}$

Output : an infinite set $G \subseteq A_i$ such that $S \not\leq_T G$

$$\Phi_{e_0}^{G_0} \neq S \vee \Phi_{e_1}^{G_1} \neq S$$

Input : a set $S \not\leq_T \emptyset$ and a 2-partition $A_0 \sqcup A_1 = \mathbb{N}$

Output : an infinite set $G \subseteq A_i$ such that $S \not\leq_T G$

$$\Phi_{e_0}^{G_0} \neq S \vee \Phi_{e_1}^{G_1} \neq S$$

The set $\left\{ \begin{array}{rl} c : c \Vdash (\exists x) & \Phi_{e_0}^{G_0}(x) \downarrow \neq S(x) \vee \Phi_{e_0}^{G_0}(x) \uparrow \\ \vee & \Phi_{e_1}^{G_1}(x) \downarrow \neq S(x) \vee \Phi_{e_1}^{G_1}(x) \uparrow \end{array} \right\}$ is dense

FIRST ATTEMPT

Given a condition $c = (F_0, F_1, X)$, suppose the formula

$$\varphi(x, n) = (\exists d \leq c) d \Vdash \Phi_{e_0}^{G_0}(x) \downarrow = n$$

is $\Sigma_1^{0,X}$ (it is not). Then the set

$$\mathcal{C} = \{(x, n) : \varphi(x, n)\}$$

is X-c.e.

FIRST ATTEMPT

$$\mathcal{C} = \{(x, n) : \varphi(x, n)\}$$

Σ_1 case

$(\exists x)(x, 1 - S(x)) \in \mathcal{C}$

Then $\exists d \leq c$ such that

$d \Vdash \Phi_{e_0}^{G_0}(x) \downarrow = 1 - S(x)$

Π_1 case

$(\exists x)(x, S(x)) \notin \mathcal{C}$

Then

$c \Vdash \Phi_{e_0}^{G_0}(x) \neq S(x)$

Impossible case

$(\forall x)(x, 1 - S(x)) \notin \mathcal{C}$

$(\forall x)(x, S(x)) \in \mathcal{C}$

Then since $\mathcal{C}$ is X-c.e

$S \leq_T X$ ↯

THE FIRST ATTEMPT FAILS

Given a condition $c = (F_0, F_1, X)$, the formula

$$\varphi(x, n) = (\exists d \leq c) d \Vdash \Phi_{e_0}^{G_0}(x) \downarrow = n$$

is too complex because it can be translated in

$$(\exists E_0 \subseteq X \cap A_0) \Phi_{e_0}^{F_0 \cup E_0}(x) \downarrow = n$$

which is $\Sigma_1^{0, A \oplus X}$ and not $\Sigma_1^{0, X}$.

IDEA: MAKE AN OVERAPPROXIMATION

"Can we find an extension for every instance of RT^1_2?"

Given a condition $c = (F_0, F_1, X)$, let $\psi(x, n)$ be the formula

$$(\forall B_0 \sqcup B_1 = \mathbb{N})(\exists i < 2)(\exists E_i \subseteq X \cap B_i)\Phi_{e_i}^{F_i \cup E_i}(x) \downarrow = n$$

$$\psi(x, n) \text{ is } \Sigma^{0,X}_1$$

Case 1: $\psi(x, n)$ holds

Letting $B_i = A_i$, there is an extension $d \leq c$ forcing

$$\Phi_{e_0}^{G_0}(x) \downarrow= n \vee \Phi_{e_1}^{G_1}(x) \downarrow= n$$

Case 2: $\psi(x, n)$ does not hold

$$(\exists B_0 \sqcup B_1 = \mathbb{N})(\forall i < 2)(\forall E_i \subseteq X \cap B_i)\Phi_{e_i}^{F_i \cup E_i}(x) \neq n$$

The condition $(F_0, F_1, X \cap B_i) \leq c$ forces

$$\Phi_{e_0}^{G_0}(x) \neq n \vee \Phi_{e_1}^{G_1}(x) \neq n$$

SECOND ATTEMPT

$$\mathcal{D} = \{(x, n) : \psi(x, n)\}$$

Σ_1 case

$(\exists x)(x, 1 - S(x)) \in \mathcal{D}$

Then $\exists d \leq c \; \exists i < 2$

$d \Vdash \Phi_{e_i}^{G_i}(x) \downarrow = 1 - S(x)$

Π_1 case

$(\exists x)(x, S(x)) \notin \mathcal{D}$

Then $\exists d \leq c \; \exists i < 2$

$d \Vdash \Phi_{e_i}^{G_i}(x) \neq S(x)$

Impossible case

$(\forall x)(x, 1 - S(x)) \notin \mathcal{D}$

$(\forall x)(x, S(x)) \in \mathcal{D}$

Then since $\mathcal{D}$ is X-c.e

$S \leq_T X \lightning$

CJS ARGUMENT

Context: We build a solution G to a P-instance X

Goal: Decide a property $\varphi(G)$.

Question: For every P-instance Y, can I find a solution G satisfying $\varphi(G)$?

If yes: In particular for $Y = X$, I can satisfy $\varphi(G)$.

If no: If no: By making G be a solution to X and Y simultaneously, I will satisfy $\neg\varphi(G)$.

Separations of Ramsey-type statements using the CJS argument often yield

tight bounds

RAMSEY'S THEOREM

Over n-tuples

$$\mathrm{RT}^{n}_{k}$$

Using k colors

RAMSEY'S THEOREM

Over n-tuples

$$\mathsf{RT}^{n}_{k,r}$$

Allows r colors

Using k colors

Fix a problem P.

A set S is P-encodable if there is an instance of P such that every solution computes S.

What sets can encode an instance of RT^n_k?

Thm (Wang)

A set is $\mathsf{RT}^n_{k,\ell}$-encodable iff it is computable for large ℓ

(whenever ℓ is at least the nth Schröder Number)

Thm (Wang)

A set is $\mathsf{RT}^n_{k,\ell}$-encodable iff it is computable for large ℓ

(whenever ℓ is at least the nth Schröder Number)

Thm (Dorais, Dzhafarov, Hirst, Mileti, Shafer)

A set is $\mathsf{RT}^n_{k,\ell}$-encodable iff it is hyperarithmetic for small ℓ

(whenever $\ell < 2^{n-1}$)

Thm (Wang)

A set is $\mathrm{RT}^n_{k,\ell}$-encodable iff it is computable for large ℓ

(whenever ℓ is at least the nth Schröder Number)

Thm (Dorais, Dzhafarov, Hirst, Mileti, Shafer)

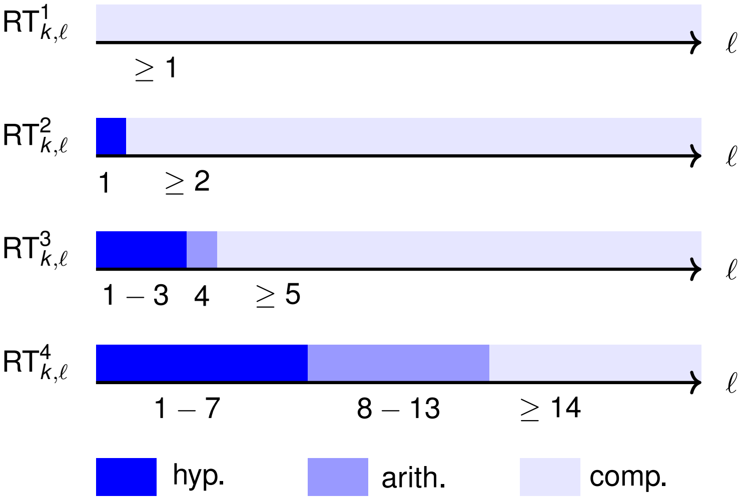
A set is $\mathrm{RT}^n_{k,\ell}$-encodable iff it is hyperarithmetic for small ℓ

(whenever $\ell < 2^{n-1}$)

Thm (Cholak, P.)

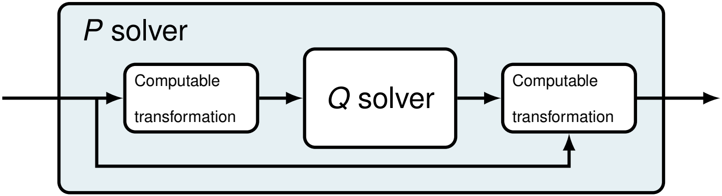
A set is $\mathrm{RT}^n_{k,\ell}$-encodable iff it is arithmetic for medium ℓ

$\mathrm{RT}^n_{k,\ell}$-ENCODABLE SETS



The CJS argument applies
to many frameworks

COMPUTABLE REDUCTION



$$\mathrm{P} \leq_c \mathrm{Q}$$

Every P-instance I computes a Q-instance J such that for every solution X to J, $X \oplus I$ computes a solution to I.

A function f is hyperimmune if it is not dominated by a computable function.

A problem P preserves ℓ among k hyperimmunities if for every k-tuple $f_1, \dots, f_k$ of hyperimmune functions and every computable P-instance I, there is a solution Y such that at least ℓ among k of the f_i are Y-hyperimmune.

Thm (P.)

RT^2_k preserves 2 among $k+1$ hyperimmunities, but not RT^2_{k+1}.

Cor (P.)

$\mathsf{RT}^2_{k+1} \not\leq_c \mathsf{RT}^2_k$.

How many applications needed to prove that
$\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \to \mathsf{RT}^2_5$?

Take an RT^2_5-instance which does not preserve 2 among 5 hyperimmune sets $A_0, \dots, A_4$.

# of apps of RT^2_2	# of i's such that A_i is hyperimmune
0	5
1	$\pi(5, 2) = 3$
2	$\pi(3, 2) = 2$
3	$\pi(2, 2) = 1$

How many applications needed to prove that
$\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \to \mathsf{RT}^2_5$?

We need at least 3 applications of RT^2_2 to obtain RT^2_5.

By a standard color blindness argument,
3 applications are sufficient.

The limits of Mathias forcing and the CJS argument

$f : [\mathbb{N}]^{n+1} \to k$ is stable if for every $\sigma \in [\mathbb{N}]^n$, $\lim_y f(\sigma, y)$ exists.

SRT^n_k : RT^n_k restricted to stable colorings.

An infinite set C is $\vec{R}$-cohesive for some sets $R_0, R_1, \dots$
if for every i, either $C \subseteq^* R_i$ or $C \subseteq^* \overline{R}_i$.

COH : Every collection of sets has a cohesive set.

$\emptyset'$-computable $\quad$ stable computable

$$\mathsf{RT}^n_k \quad \Leftrightarrow \quad \mathsf{RT}^{n+1}_k$$

$\emptyset'$-computable		stable computable
RT^n_k	$\Leftrightarrow$	RT^{n+1}_k
"Every Δ^0_2 set has an infinite subset or cosubset"	$\Leftrightarrow$	SRT^2_2

$$\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \leftrightarrow \mathsf{COH} \wedge \mathsf{SRT}^2_2.$$

Given $f : [\mathbb{N}]^2 \to 2$, define $\langle R_x : x \in \mathbb{N} \rangle$ by

$$R_x = \{y : f(x, y) = 1\}$$

By COH, there is an $\vec{R}$-cohesive set C.

$f : [C]^2 \to 2$ is an instance of SRT^2_2

$$\mathsf{RCA}_0 \nvdash \mathsf{COH} \rightarrow \mathsf{SRT}^2_2$$

(Hirschfeldt, Jocksuch, Kjos-Hanssen, Lempp, and Slaman)

By preserving $\mathcal{W} = \{X :$
X does not compute an f-homogeneous set $\}$
using a computable Mathias forcing.

$$\mathrm{RCA}_0 \nvdash \mathrm{SRT}^2_2 \rightarrow \mathrm{COH}$$

(Chong, Slaman and Yang)

Using the CJS argument in a non-standard model containing only low sets.

Turing ideal $\mathcal{M}$

- $(\forall X \in \mathcal{M})(\forall Y \leq_T X)[Y \in \mathcal{M}]$
- $(\forall X, Y \in \mathcal{M})[X \oplus Y \in \mathcal{M}]$

Examples

- $\{X : X \text{ is computable }\}$
- $\{X : X \leq_T A \wedge X \leq_T B\}$ for some sets A and B

Let $\mathcal{M}$ be a Turing ideal and P, Q be problems.

Satisfaction

$$\mathcal{M} \models \mathsf{P}$$

if every P-instance in $\mathcal{M}$ has a solution in $\mathcal{M}$.

Computable entailment

$$\mathsf{P} \models_c \mathsf{Q}$$

if every Turing ideal satisfying P satisfies Q.

Does $\mathsf{SRT}^2_2 \models_c \mathsf{COH}$?

(Hirschfeldt)

The CJS argument applied to RT^1_2 yields solutions to COH.

Does $\mathsf{COH} \leq_c \mathsf{SRT}^2_2$?

(Dzhafarov)

Have we found the right framework?

Can Mathias forcing and the CJS argument answer all the Ramsey-type questions?

The CJS argument applied to RT^1_2 yields solutions to COH.

Fix a computable sequence of sets $R_0, R_1, \dots$

Is there a set X, such that every infinite set $H \subseteq X$ or $H \subseteq \overline{X}$ computes an $\vec{R}$-cohesive set?

A set X is high if $X' \geq_T \emptyset''$.

Is there a set X, such that every infinite set $H \subseteq X$ or $H \subseteq \overline{X}$ is high?

If yes, then COH $\leq_{oc}$ RT^1_2.

If no, well, this is still interesting *per se*.

A set S is P-jump-encodable if there is an instance of P such that the jump of every solution computes S.

Are the RT^1_2-jump-encodable sets precisely the $\emptyset'$-computable ones?

CONCLUSION

We have a minimalistic framework which answers accurately many questions about Ramsey's theorem.

This can be taken as evidence that we have found the right framework.

Does the COH vs SRT^2_2 question reveal the limits of the framework?

REFERENCES

Peter A. Cholak, Carl G. Jockusch, and Theodore A. Slaman.
On the strength of Ramsey's theorem for pairs.
Journal of Symbolic Logic, 66(01):1–55, 2001.

Carl G. Jockusch.
Ramsey's theorem and recursion theory.
Journal of Symbolic Logic, 37(2):268–280, 1972.

Ludovic Patey.
The reverse mathematics of Ramsey-type theorems.
PhD thesis, Université Paris Diderot, 2016.

Wei Wang.
Some logically weak Ramseyan theorems.
Advances in Mathematics, 261:1–25, 2014.