The strength of Ramsey's theorem under reducibilities

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STRENGTH OF A THEOREM

Some theorems are more effective than others.

Theorem (Intermediate value theorem)

For every continuous function f over $[a,b]$ and every $y \in [f(a), f(b)]$, there is some $x \in [a,b]$ such that $f(x) = y$.

Theorem (König's lemma)

Every infinite, finitely branching tree has an infinite path.

STRENGTH OF A THEOREM

Provability strength

- Reverse mathematics
- Intuitionistic reverse mathematics

Computational strength

- Computable reducibility
- Uniform reducibility

Provability approach

REVERSE MATHEMATICS

Goal

Determine which axioms are required to prove ordinary theorems in reverse mathematics.

- Simpler proofs
- More insights

Subsystems of second-order arithmetic.

BASE THEORY RCA_0

- Basic Peano axioms
- Σ^0_1 induction scheme

$$(\varphi(0) \wedge \forall n.(\varphi(n) \to \varphi(n+1))) \to \forall n.\varphi(n)$$

where $\varphi(n)$ is any Σ^0_1 formula of L_2

- Δ^0_1 comprehension scheme

$$\forall n(\varphi(n) \leftrightarrow \psi(n)) \to \exists X.\forall n.(x \in X \leftrightarrow \varphi(n))$$

where $\varphi(n)$ is any Σ^0_1 formula of L_2 in which X does not occur freely and $\psi(n)$ is any Π^0_1 formula of L_2.

HOW TO THINK ABOUT RCA_0 ?

RCA_0 captures computable mathematics

RCA_0 has model $\mathcal{M} = \{\omega, S, <, +, \cdot\}$ where

- ω is the set of the standard integers
- $S = \{X \in 2^\omega : X$ is computable $\}$ is the second-order part

Computational approach

THEOREMS AS PROBLEMS

Many theorems P are of the form

$$(\forall X)[\Phi(X) \to (\exists Y)\Psi(X, Y)]$$

where Φ and Ψ are arithmetic formulas.

We may think of P as a class of problems.

- An X such that $\Phi(X)$ holds is an instance.
- A Y such that $\Psi(X, Y)$ holds is a solution to X.

THEOREMS AS PROBLEMS

Examples:

- (König's lemma)
  Every infinite, finitely branching tree has an infinite path.
- (Ramsey's theorem)
  Every k-coloring has an infinite monochromatic subset.
- (The atomic model theorem)
  Every complete atomic theory has an atomic model.
- ...

COMPUTABLE REDUCIBILITY

Definition (Computable reducibility)

A theorem P *is computably reducible to a theorem* Q *if every* P*-instance* I *computes a* Q*-instance* J *such that for every solution* X *to* J, $X \oplus I$ *computes a solution to* I.

Intuition:

If $\mathsf{P} \leq_c \mathsf{Q}$ then solving Q is harder than solving P.

PROVABILITY VS COMPUTATIONAL APPROACH

If we forget induction,

$$\mathsf{P} \leq_c \mathsf{Q}$$

can be seen as

$$\mathsf{RCA}_0 \vdash \mathsf{Q} \rightarrow \mathsf{P}$$

where only one application of Q is allowed.

Ramsey's theorem

RAMSEY'S THEORY

Given some size s, every sufficiently large collection of objects has a sub-collection of size s, whose objects satisfy some structural properties.

RAMSEY'S THEOREM

Definition

Given a coloring $f : [\mathbb{N}]^n \to k$, a set H is f-homogeneous if there exists a color $i < k$ such that $f([H]^n) = i$.

Definition (Ramsey's theorem)

Every coloring $f : [\mathbb{N}]^n \to k$ has an infinite f-homogeneous set.

RAMSEY'S THEOREM

Over k-tuples

$$\mathsf{RT}^{k}_{n}$$

Using n colors

RAMSEY'S THEOREM

Fix the number of colors n.

RAMSEY'S THEOREM FOR k-TUPLES

Theorem (Jockusch, 1972)
Every computable coloring $f : [\mathbb{N}]^k \to n$ has a Π^0_k infinite f-homogeneous set.

Theorem (Jockusch, 1972)
For every $k \geq 3$, there is a computable coloring $f : [\mathbb{N}]^k \to n$ such that every infinite f-homogeneous set computes $\emptyset^{(k-2)}$.

RAMSEY'S THEOREM FOR k-TUPLES

Theorem (Simpson, 2009)

For each $k_1, k_2 \geq 3$, $\mathsf{RCA}_0 \vdash \mathsf{RT}^{k_1}_n \leftrightarrow \mathsf{RT}^{k_2}_n$.

What about RT^2_n ?

RAMSEY'S THEOREM FOR PAIRS

Theorem (Seetapun, 1995)

For every computable coloring $f : [\mathbb{N}]^2 \to n$ *and every non-computable set* C*, there is an infinite* f*-homogeneous set* $H \not\geq_T C$.

Corollary

RT^2_n *does not imply* RT^3_n *over* RCA_0.

HOW MANY APPLICATIONS?

When $3 \leq k_1 < k_2$, the proof of

$$\mathsf{RCA_0} \vdash \mathsf{RT}^{k_1}_n \to \mathsf{RT}^{k_2}_n$$

involves multiple applications of $\mathsf{RT}^{k_1}_n$.

How many applications of $\mathsf{RT}^{k_1}_n$ are necessary?

HOW MANY APPLICATIONS?

Theorem (Jockusch, 1972)

For every $k \geq 2$, *there is a computable coloring* $f : [\mathbb{N}]^k \to n$ *with no* Σ^0_k *infinite* f*-homogeneous set.*

Corollary

For every $k \geq 2$, $\mathsf{RT}^k_n \not\leq_c \mathsf{RT}^{k+1}_n$.

At least 2 applications of RT^k_n are necessary to prove RT^{k+1}_n.

HOW MANY APPLICATIONS?

Theorem (Cholak, Jockusch, Slaman, 2001)

For every $k \geq 2$, every set $P \gg \emptyset^{(k-1)}$, and every computable coloring $f : [\mathbb{N}]^k \to n$, there is an infinite f-homogeneous set H such that $H' \leq_T P$.

- At most 3 applications of RT^3_n are necessary to prove RT^4_n
- Exactly 2 applications of RT^k_n are necessary to prove RT^{k+1}_n whenever $k \geq 4$.

SUMMARY FOR A FIXED n

$$\begin{array}{ccc} & & \vdots \\ & & \downarrow \\ & & \mathsf{RT}^4_n \\ & & \downarrow \\ \mathsf{RT}^k_n, k \geq 3 & & \mathsf{RT}^3_n \\ \downarrow & & \downarrow \\ \mathsf{RT}^2_n & & \mathsf{RT}^2_n \\ & & \\ \text{Over } \mathsf{RCA}_0 & & \text{Over } \leq_c \end{array}$$

RAMSEY'S THEOREM

Fix the size of tuples k.

RAMSEY'S THEOREM

Theorem (Folklore)

For every $n, m \geq 2$*,* $\mathsf{RCA}_0 \vdash \mathsf{RT}^k_n \leftrightarrow \mathsf{RT}^k_m$

Proof for $m = n^2$.

- Take a coloring $f : [\mathbb{N}]^k \to n^2$
- Define $g : [\mathbb{N}]^k \to n$ by merging colors by blocks of size n
- Apply RT^k_n to g to obtain H such that $|f([H]^2)| \leq n$.
- Apply again RT^k_n to f restricted to H.

$\square$

HOW MANY APPLICATIONS?

Theorem (Patey)

Fix some $n > m \geq 2$ and n sets $B_0, \ldots, B_{n-1}$ whose complements are hyperimmune. For every m-partition $A_0 \cup \cdots \cup A_{m-1} = \mathbb{N}$, there exists an infinite subset H of some A_i and a pair $j_0 < j_1 < n$ such that every infinite H-computable set intersects both B_{j_0} and B_{j_1}.

HOW MANY APPLICATIONS?

Theorem

For every $n > m \geq 2$, $\mathsf{RT}^2_n \not\leq_c \mathsf{RT}^2_m$.

Proof (Part I).

- Define a Δ^0_2 partition $B_0 \cup \cdots \cup B_{n-1} = \mathbb{N}$ such that the $\overline{B}$'s are hyperimmune.
- Consider its Δ^0_2 approximation function as a computable instance of RT^2_n.

$\square$

HOW MANY APPLICATIONS?

Theorem

For every $n > m \geq 2$, $\mathsf{RT}^2_n \not\leq_c \mathsf{RT}^2_m$.

Proof (Part II).

- Fix computable instance $f : [\mathbb{N}]^2 \to m$ of RT^2_m.
- Construct a p-cohesive set C such that the $\overline{B}$'s are hyperimmune relative to C.
- Define $\tilde{f} : \mathbb{N} \to m$ by $\tilde{f}(x) = \lim_{s \in C} f(x, s)$
- Apply previous theorem to obtain an infinite $\tilde{f}$-homogeneous set H such that $H \oplus C$ does not compute an infinite set homogeneous for the B's.

$\square$

HOW MANY APPLICATIONS?

Theorem (Patey)

For every $n > m \geq 2$, $\mathsf{RT}^k_n \not\leq_c \mathsf{RT}^k_m$.

Proof.

By induction over $k \geq 2$ using prehomogeneous sets. $\square$

SUMMARY FOR A FIXED k

$$\begin{array}{cc} & \vdots \\ & \downarrow \\ & \mathsf{RT}^k_4 \\ & \downarrow \\ & \mathsf{RT}^k_3 \\ & \downarrow \\ \mathsf{RT}^k_n,\ n \geq 2 & \mathsf{RT}^k_2 \end{array}$$

Over RCA_0 Over $\leq_c$

RAMSEY'S THEOREM

Over k-tuples

$$\mathsf{RT}^k_n$$

Using n colors

RAMSEY'S THEOREM

Over k-tuples

$$\mathrm{RT}^{k}_{n,r}$$

Allows r colors

Using n colors

THIN SET THEOREM

$$\mathsf{TS}^k_n \qquad \mathsf{RT}^k_{n,n-1}$$

ALLOWING MORE COLORS

Theorem (Wang, 2014)

Fix some k and some sufficiently large n. For every computable instance f of TS^k_n and every non-computable set C, there is an infinite solution to f which does not compute C.

Corollary

For every k and sufficiently large n, TS^k_n does not imply RT^3_2 over RCA_0.

Allowing more colors

Theorem (Dorais, Dzhafarov, Hirst, Mileti, Shafer, 2015)
$\mathsf{RCA}_0 \vdash \mathsf{TS}^{ks+1}_{n^s} \to \mathsf{TS}^{k+1}_{n}$

Theorem (Dorais, Dzhafarov, Hirst, Mileti, Shafer, 2015)
$\mathsf{RCA}_0 \vdash \mathsf{TS}^{k+2}_{2^k} \to \mathsf{TS}^{3}_{2}$

ALLOWING MORE COLORS

Theorem (Patey)

For every $n \geq 2$, TS^2_{n+1} does not imply TS^2_n over RCA_0.

Theorem (Patey)

Fix some $m \geq 2$. For every k and sufficiently large n, TS^k_n does not imply TS^2_m over RCA_0.

SUMMARY FOR $k = 2$

$$\begin{gathered} \mathsf{RT}^2_2 \\ \downarrow \\ \mathsf{TS}^2_3 \\ \downarrow \\ \mathsf{TS}^2_4 \\ \vdots \\ \downarrow \end{gathered}$$

Over RCA_0

CONCLUSION

- Computable reducibility gives a more fine-grained analysis than reverse mathematics.
- Ramsey's theorem is not robust for computable reducibility.
- Changing the number of allowed colors has a great impact on the strength of Ramsey's theorem.

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Questions

Thank you for listening!