

# The strength of Ramsey's theorem under reducibilities

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# STRENGTH OF A THEOREM

Some theorems are more **effective** than others.

Theorem (Intermediate value theorem)

*For every continuous function  $f$  over  $[a, b]$  and every  $y \in [f(a), f(b)]$ , there is some  $x \in [a, b]$  such that  $f(x) = y$ .*

Theorem (König's lemma)

*Every infinite, finitely branching tree has an infinite path.*

# STRENGTH OF A THEOREM

## Provability strength

- ▶ Reverse mathematics
- ▶ Intuitionistic reverse mathematics

## Computational strength

- ▶ Computable reducibility
- ▶ Uniform reducibility

# Provability approach

# REVERSE MATHEMATICS

## Goal

Determine which axioms are required to prove **ordinary** theorems in reverse mathematics.

- ▶ Simpler proofs
- ▶ More insights

Subsystems of second-order arithmetic.

# BASE THEORY $\text{RCA}_0$

- ▶ Basic Peano axioms
- ▶  $\Sigma_1^0$  induction scheme

$$(\varphi(0) \wedge \forall n.(\varphi(n) \rightarrow \varphi(n+1))) \rightarrow \forall n.\varphi(n)$$

where  $\varphi(n)$  is any  $\Sigma_1^0$  formula of  $L_2$

- ▶  $\Delta_1^0$  comprehension scheme

$$\forall n(\varphi(n) \leftrightarrow \psi(n)) \rightarrow \exists X.\forall n.(x \in X \leftrightarrow \varphi(n))$$

where  $\varphi(n)$  is any  $\Sigma_1^0$  formula of  $L_2$  in which  $X$  does not occur freely and  $\psi(n)$  is any  $\Pi_1^0$  formula of  $L_2$ .

# HOW TO THINK ABOUT $\text{RCA}_0$ ?

$\text{RCA}_0$  captures **computable** mathematics

$\text{RCA}_0$  has model  $\mathcal{M} = \{\omega, S, <, +, \cdot\}$  where

- ▶  $\omega$  is the set of the standard integers
- ▶  $S = \{X \in 2^\omega : X \text{ is computable}\}$  is the second-order part

# Computational approach



# THEOREMS AS PROBLEMS

Many theorems  $\mathbf{P}$  are of the form

$$(\forall X)[\Phi(X) \rightarrow (\exists Y)\Psi(X, Y)]$$

where  $\Phi$  and  $\Psi$  are arithmetic formulas.

We may think of  $\mathbf{P}$  as a class of **problems**.

- ▶ An  $X$  such that  $\Phi(X)$  holds is an **instance**.
- ▶ A  $Y$  such that  $\Psi(X, Y)$  holds is a **solution** to  $X$ .

# THEOREMS AS PROBLEMS

Examples:

- ▶ (König's lemma)  
Every **infinite, finitely branching tree** has an **infinite path**.
- ▶ (Ramsey's theorem)  
Every  **$k$ -coloring** has an **infinite monochromatic subset**.
- ▶ (The atomic model theorem)  
Every **complete atomic theory** has an **atomic model**.
- ▶ ...

# COMPUTABLE REDUCIBILITY

## Definition (Computable reducibility)

A theorem  $P$  is *computably reducible* to a theorem  $Q$  if every  $P$ -instance  $I$  computes a  $Q$ -instance  $J$  such that for every solution  $X$  to  $J$ ,  $X \oplus I$  computes a solution to  $I$ .

Intuition:

If  $P \leq_c Q$  then solving  $Q$  is *harder* than solving  $P$ .

# PROVABILITY VS COMPUTATIONAL APPROACH

If we forget induction,

$$P \leq_c Q$$

can be seen as

$$\text{RCA}_0 \vdash Q \rightarrow P$$

where only one application of  $Q$  is allowed.

# Ramsey's theorem

# RAMSEY'S THEORY

Given some **size**  $s$ , every **sufficiently large** collection of objects has a sub-collection of size  $s$ , whose objects satisfy some **structural properties**.

# RAMSEY'S THEOREM

## Definition

Given a coloring  $f : [\mathbb{N}]^n \rightarrow k$ , a set  $H$  is *f-homogeneous* if there exists a color  $i < k$  such that  $f([H]^n) = i$ .

## Definition (Ramsey's theorem)

Every coloring  $f : [\mathbb{N}]^n \rightarrow k$  has an infinite *f-homogeneous* set.

# RAMSEY'S THEOREM

Over  $k$ -tuples

RT  $k$   
 $n$

Using  $n$  colors



# RAMSEY'S THEOREM

Fix the number of colors  $n$ .

# RAMSEY'S THEOREM FOR $k$ -TUPLES

Theorem (Jockusch, 1972)

*Every computable coloring  $f : [\mathbb{N}]^k \rightarrow n$  has a  $\Pi_k^0$  infinite  $f$ -homogeneous set.*

Theorem (Jockusch, 1972)

*For every  $k \geq 3$ , there is a computable coloring  $f : [\mathbb{N}]^k \rightarrow n$  such that every infinite  $f$ -homogeneous set computes  $\emptyset^{(k-2)}$ .*

# RAMSEY'S THEOREM FOR $k$ -TUPLES

Theorem (Simpson, 2009)

For each  $k_1, k_2 \geq 3$ ,  $\text{RCA}_0 \vdash \text{RT}_n^{k_1} \leftrightarrow \text{RT}_n^{k_2}$ .

What about  $\text{RT}_n^2$  ?

# RAMSEY'S THEOREM FOR PAIRS

Theorem (Seetapun, 1995)

*For every computable coloring  $f : [\mathbb{N}]^2 \rightarrow n$  and every non-computable set  $C$ , there is an infinite  $f$ -homogeneous set  $H \not\leq_T C$ .*

Corollary

*$\text{RT}_n^2$  does not imply  $\text{RT}_n^3$  over  $\text{RCA}_0$ .*

# HOW MANY APPLICATIONS?

When  $3 \leq k_1 < k_2$ , the proof of

$$\text{RCA}_0 \vdash \text{RT}_n^{k_1} \rightarrow \text{RT}_n^{k_2}$$

involves multiple applications of  $\text{RT}_n^{k_1}$ .

How many applications of  $\text{RT}_n^{k_1}$  are necessary?

# HOW MANY APPLICATIONS?

Theorem (Jockusch, 1972)

*For every  $k \geq 2$ , there is a computable coloring  $f : [\mathbb{N}]^k \rightarrow n$  with no  $\Sigma_k^0$  infinite  $f$ -homogeneous set.*

Corollary

*For every  $k \geq 2$ ,  $\text{RT}_n^k \not\leq_c \text{RT}_n^{k+1}$ .*

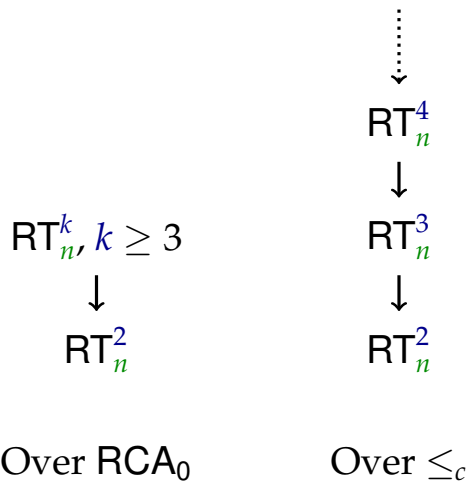
At least 2 applications of  $\text{RT}_n^k$  are necessary to prove  $\text{RT}_n^{k+1}$ .

# HOW MANY APPLICATIONS?

Theorem (Cholak, Jockusch, Slaman, 2001)

*For every  $k \geq 2$ , every set  $P \gg \emptyset^{(k-1)}$ , and every computable coloring  $f : [\mathbb{N}]^k \rightarrow n$ , there is an infinite  $f$ -homogeneous set  $H$  such that  $H' \leq_T P$ .*

- ▶ At most 3 applications of  $\text{RT}_n^3$  are necessary to prove  $\text{RT}_n^4$
- ▶ Exactly 2 applications of  $\text{RT}_n^k$  are necessary to prove  $\text{RT}_n^{k+1}$  whenever  $k \geq 4$ .

SUMMARY FOR A FIXED  $n$ 



# RAMSEY'S THEOREM

Fix the size of tuples  $k$ .

# RAMSEY'S THEOREM

## Theorem (Folklore)

For every  $n, m \geq 2$ ,  $\text{RCA}_0 \vdash \text{RT}_n^k \leftrightarrow \text{RT}_m^k$

Proof for  $m = n^2$ .

- ▶ Take a coloring  $f : [\mathbb{N}]^k \rightarrow n^2$
- ▶ Define  $g : [\mathbb{N}]^k \rightarrow n$  by merging colors by blocks of size  $n$
- ▶ Apply  $\text{RT}_n^k$  to  $g$  to obtain  $H$  such that  $|f([H]^2)| \leq n$ .
- ▶ Apply again  $\text{RT}_n^k$  to  $f$  restricted to  $H$ .

□

# HOW MANY APPLICATIONS?

## Theorem (Patey)

*Fix some  $n > m \geq 2$  and  $n$  sets  $B_0, \dots, B_{n-1}$  whose complements are hyperimmune. For every  $m$ -partition  $A_0 \cup \dots \cup A_{m-1} = \mathbb{N}$ , there exists an infinite subset  $H$  of some  $A_i$  and a pair  $j_0 < j_1 < n$  such that every infinite  $H$ -computable set intersects both  $B_{j_0}$  and  $B_{j_1}$ .*

# HOW MANY APPLICATIONS?

## Theorem

For every  $n > m \geq 2$ ,  $\text{RT}_n^2 \not\leq_c \text{RT}_m^2$ .

## Proof (Part I).

- ▶ Define a  $\Delta_2^0$  partition  $B_0 \cup \dots \cup B_{n-1} = \mathbb{N}$  such that the  $\bar{B}$ 's are **hyperimmune**.
- ▶ Consider its  $\Delta_2^0$  approximation function as a computable instance of  $\text{RT}_n^2$ .

□

# HOW MANY APPLICATIONS?

## Theorem

For every  $n > m \geq 2$ ,  $\text{RT}_n^2 \not\leq_c \text{RT}_m^2$ .

Proof (Part II).

- ▶ Fix computable instance  $f : [\mathbb{N}]^2 \rightarrow m$  of  $\text{RT}_m^2$ .
- ▶ Construct a **p-cohesive** set  $C$  such that the  $\bar{B}'$ 's are hyperimmune relative to  $C$ .
- ▶ Define  $\tilde{f} : \mathbb{N} \rightarrow m$  by  $\tilde{f}(x) = \lim_{s \in C} f(x, s)$
- ▶ Apply previous theorem to obtain an infinite  $\tilde{f}$ -homogeneous set  $H$  such that  $H \oplus C$  does not compute an infinite set homogeneous for the  $B'$ 's.

□

# HOW MANY APPLICATIONS?

Theorem (Patey)

For every  $n > m \geq 2$ ,  $\text{RT}_n^k \not\leq_c \text{RT}_m^k$ .

Proof.

By induction over  $k \geq 2$  using **prehomogeneous** sets. □

SUMMARY FOR A FIXED  $k$ 

$$RT_n^k, n \geq 2$$

Over  $\text{RCA}_0$

$$\begin{array}{c} \vdots \\ \downarrow \\ RT_4^k \\ \downarrow \\ RT_3^k \\ \downarrow \\ RT_2^k \end{array}$$

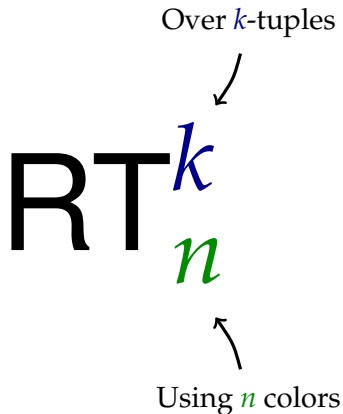
Over  $\leq_c$

# RAMSEY'S THEOREM

Over  $k$ -tuples

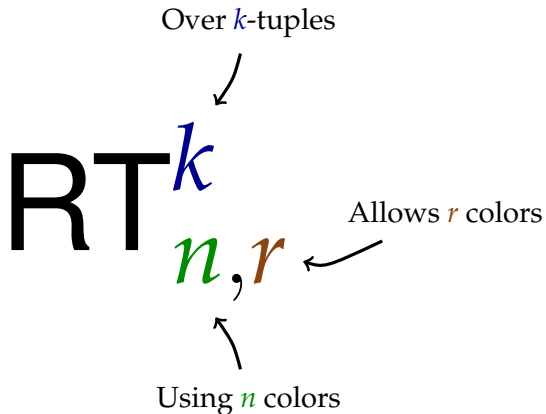
$RT^k_n$

Using  $n$  colors





## RAMSEY'S THEOREM



## THIN SET THEOREM

 $TS_n^k$  $RT_{n,n-1}^k$

# ALLOWING MORE COLORS

## Theorem (Wang, 2014)

Fix some  $k$  and some *sufficiently large*  $n$ . For every computable instance  $f$  of  $\text{TS}_n^k$  and every non-computable set  $C$ , there is an infinite solution to  $f$  which does not compute  $C$ .

## Corollary

For every  $k$  and sufficiently large  $n$ ,  $\text{TS}_n^k$  does not imply  $\text{RT}_2^3$  over  $\text{RCA}_0$ .

# ALLOWING MORE COLORS

Theorem (Dorais, Dzhafarov, Hirst, Mileti, Shafer, 2015)

$$\text{RCA}_0 \vdash \text{TS}_{n^s}^{ks+1} \rightarrow \text{TS}_n^{k+1}$$

Theorem (Dorais, Dzhafarov, Hirst, Mileti, Shafer, 2015)

$$\text{RCA}_0 \vdash \text{TS}_{2^k}^{k+2} \rightarrow \text{TS}_2^3$$

# ALLOWING MORE COLORS

## Theorem (Patey)

*For every  $n \geq 2$ ,  $\text{TS}_{n+1}^2$  does not imply  $\text{TS}_n^2$  over  $\text{RCA}_0$ .*

## Theorem (Patey)

*Fix some  $m \geq 2$ . For every  $k$  and sufficiently large  $n$ ,  $\text{TS}_n^k$  does not imply  $\text{TS}_m^2$  over  $\text{RCA}_0$ .*

SUMMARY FOR  $k = 2$  $RT_2^2$  $TS_3^2$  $TS_4^2$ Over  $RCA_0$

# CONCLUSION

- ▶ Computable reducibility gives a more fine-grained analysis than reverse mathematics.
- ▶ Ramsey's theorem is not robust for computable reducibility.
- ▶ Changing the number of allowed colors has a great impact on the strength of Ramsey's theorem.

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# QUESTIONS

Thank you for listening!