The strength of Ramsey's theorem under reducibilities

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STRENGTH OF A THEOREM

Some theorems are more effective than others.

Theorem (Intermediate value theorem)

For every continuous function f over $[a, b]$ and every $y \in [f(a), f(b)]$, there is some $x \in [a, b]$ such that $f(x) = y$.

Theorem (König's lemma)

Every infinite, finitely branching tree has an infinite path.

STRENGTH OF A THEOREM

Provability strength

- Reverse mathematics
- Intuitionistic reverse mathematics

Computational strength

- Computable reducibility
- Uniform reducibility

Provability approach

REVERSE MATHEMATICS

Goal

Determine which axioms are required to prove ordinary theorems in reverse mathematics.

- Simpler proofs
- More insights

Subsystems of second-order arithmetic.

BASE THEORY RCA_0

- Basic Peano axioms
- Σ^0_1 induction scheme

$$(\varphi(0) \wedge \forall n.(\varphi(n) \rightarrow \varphi(n+1))) \rightarrow \forall n.\varphi(n)$$

where $\varphi(n)$ is any Σ^0_1 formula of L_2

- Δ^0_1 comprehension scheme

$$\forall n(\varphi(n) \leftrightarrow \psi(n)) \rightarrow \exists X.\forall n.(n \in X \leftrightarrow \varphi(n))$$

where $\varphi(n)$ is any Σ^0_1 formula of L_2 in which X does not occur freely and $\psi(n)$ is any Π^0_1 formula of L_2.

STANDARD MODELS OF RCA_0

An ω-structure is a structure $\mathcal{M} = \{\omega, \mathcal{S}, <, +, \cdot\}$ where

(i) ω is the set of standard natural numbers

(ii) $<$ is the natural order

(iii) $+$ and $\cdot$ are the standard operations over natural numbers

(iv) $\mathcal{S} \subseteq \mathcal{P}(\omega)$

An ω-structure is fully specified by its second-order part $\mathcal{S}$.

STANDARD MODELS OF RCA_0

Definition (Turing ideal)

A Turing ideal $\mathcal{I}$ is a collection of subsets of ω which is closed under

(i) the Turing reduction: $(\forall X \in \mathcal{I})(\forall Y \leq_T X)[Y \in \mathcal{I}]$

(ii) the effective join: $(\forall X, Y \in \mathcal{I})[X \oplus Y \in \mathcal{I}]$.

STANDARD MODELS OF RCA_0

Fix an ω-structure $\mathcal{M} = \{\omega, \mathcal{S}, <, +, \cdot\}$.

$$\mathcal{M} \models \mathsf{RCA}_0 \qquad \equiv \qquad \mathcal{S} \text{ is a Turing ideal.}$$

HOW TO THINK ABOUT RCA_0 ?

RCA_0 captures computable mathematics

RCA_0 a minimal ω-model $\mathcal{M} = \{\omega, \mathcal{I}, <, +, \cdot\}$
where $\mathcal{I}$ is the set of all computable subsets of ω.

Computational approach

THEOREMS AS PROBLEMS

Many theorems P are of the form

$$(\forall X)[\Phi(X) \to (\exists Y)\Psi(X, Y)]$$

where Φ and Ψ are arithmetic formulas.

We may think of P as a class of problems.

- An X such that $\Phi(X)$ holds is an instance.
- A Y such that $\Psi(X, Y)$ holds is a solution to X.

THEOREMS AS PROBLEMS

Examples:

- (König's lemma)
  Every infinite, finitely branching tree has an infinite path.
- (Ramsey's theorem)
  Every k-coloring has an infinite monochromatic subset.
- (The atomic model theorem)
  Every complete atomic theory has an atomic model.
- ...

COMPUTABLE REDUCIBILITY

Definition (Computable reducibility)

A theorem P *is computably reducible to a theorem* Q *if every* P*-instance* I *computes a* Q*-instance* J *such that for every solution* X *to* J*,* $X \oplus I$ *computes a solution to* I*.*

Intuition:

If $\mathsf{P} \leq_c \mathsf{Q}$ then solving Q is harder than solving P.

COMPUTABLE REDUCIBILITY

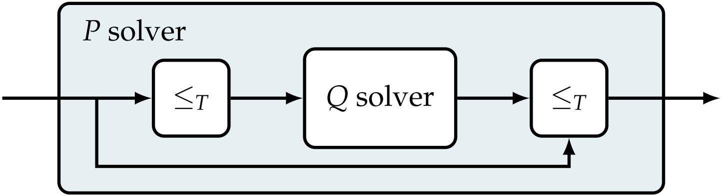

Figure: Computable reducibility

PROVABILITY VS COMPUTATIONAL APPROACH

If we forget induction,

$$\mathsf{P} \leq_c \mathsf{Q}$$

can be seen as

$$\mathsf{RCA}_0 \vdash \mathsf{Q} \rightarrow \mathsf{P}$$

where only one application of Q is allowed.

Ramsey's theorem

RAMSEY'S THEORY

Given some size s, every sufficiently large collection of objects has a sub-collection of size s, whose objects satisfy some structural properties.

Ramsey's theorem

Definition

Given a coloring $f : [\mathbb{N}]^n \to k$*, a set* H *is* *f-homogeneous* *if there exists a color* $i < k$ *such that* $f([H]^n) = i$*.*

Definition (Ramsey's theorem)

Every coloring $f : [\mathbb{N}]^n \to k$ *has an infinite* f*-homogeneous set.*

RAMSEY'S THEOREM

Over n-tuples

$$\mathsf{RT}^n_k$$

Using k colors

RAMSEY'S THEOREM

Fix the number of colors k.

RAMSEY'S THEOREM FOR n-TUPLES

Theorem (Jockusch, 1972)

Every computable coloring $f : [\mathbb{N}]^n \to k$ has a Π^0_n infinite f-homogeneous set.

Theorem (Jockusch, 1972)

For every $n \geq 3$, there is a computable coloring $f : [\mathbb{N}]^n \to k$ such that every infinite f-homogeneous set computes $\emptyset^{(n-2)}$.

RAMSEY'S THEOREM FOR n-TUPLES

Theorem (Simpson, 2009)

For each $n, m \geq 3$, $\mathsf{RCA}_0 \vdash \mathsf{RT}^n_k \leftrightarrow \mathsf{RT}^m_k$.

What about RT^2_k ?

RAMSEY'S THEOREM FOR PAIRS

Theorem (Seetapun, 1995)

For every computable coloring $f : [\mathbb{N}]^2 \to k$ *and every non-computable set* C*, there is an infinite* f*-homogeneous set* $H \not\geq_T C$*.*

Corollary

RT^2_k *does not imply* RT^3_k *over* RCA_0.

HOW MANY APPLICATIONS?

When $3 \leq m < n$, the proof of

$$\mathsf{RCA}_0 \vdash \mathsf{RT}^m_k \rightarrow \mathsf{RT}^n_k$$

involves multiple applications of RT^m_k.

How many applications of RT^m_k are necessary?

How many applications?

Theorem (Jockusch, 1972)

For every $n \geq 2$, there is a computable coloring $f : [\mathbb{N}]^n \to k$ with no Σ^0_n infinite f-homogeneous set.

Corollary

For every $n \geq 2$, $\mathsf{RT}^n_k \not\leq_c \mathsf{RT}^{n+1}_k$.

At least 2 applications of RT^n_k are necessary to prove RT^{n+1}_k.

HOW MANY APPLICATIONS?

Theorem (Cholak, Jockusch, Slaman, 2001)

For every $n \geq 2$, every set $P \gg \emptyset^{(n-1)}$, and every computable coloring $f : [\mathbb{N}]^n \to k$, there is an infinite f-homogeneous set H such that $H' \leq_T P$.

- At most 3 applications of RT^3_k are necessary to prove RT^4_k
- Exactly 2 applications of RT^n_k are necessary to prove RT^{n+1}_k whenever $n \geq 4$.

SUMMARY FOR A FIXED k

$$
\begin{array}{ccc}
 & & \vdots \\
 & & \downarrow \\
 & & \mathsf{RT}^4_k \\
 & & \downarrow \\
\mathsf{RT}^n_k,\ n \geq 3 & & \mathsf{RT}^3_k \\
\downarrow & & \downarrow \\
\mathsf{RT}^2_k & & \mathsf{RT}^2_k
\end{array}
$$

Over RCA_0 Over $\leq_c$

RAMSEY'S THEOREM

Fix the size of tuples n.

RAMSEY'S THEOREM

Theorem (Folklore)

For every $k, \ell \geq 2$, $\mathsf{RCA}_0 \vdash \mathsf{RT}^n_k \leftrightarrow \mathsf{RT}^n_\ell$

Proof for $k = \ell^2$.

- Take a coloring $f : [\mathbb{N}]^n \to \ell^2$
- Define $g : [\mathbb{N}]^n \to \ell$ by merging colors by blocks of size ℓ
- Apply RT^n_ℓ to g to obtain H such that $|f([H]^2)| \leq \ell$.
- Apply again RT^n_ℓ to f restricted to H.

$\square$

HOW MANY APPLICATIONS?

Theorem (P.)

For every $k > \ell \geq 2$, $\mathsf{RT}^n_k \not\leq_c \mathsf{RT}^n_\ell$.

Theorem (P.)

For every $k > \ell \geq 2$, there is a Δ^0_n partition $A_0 \cup \cdots \cup A_{k-1} = \mathbb{N}$ such that every computable RT^n_ℓ-instance has a homogeneous set which computes no infinite subset of one of the A's.

A hard Δ^0_2 partition

Definition
A function f is Y-hyperimmune if f is not dominated by any Y-computable function. A set X is Y-hyperimmune if its principal function p_X is.

If $\overline{X}$ is Y-hyperimmune, then every infinite Y-computable set intersects X.

A HARD Δ^0_2 PARTITION

Lemma (Folklore)

This is a Δ^0_2 partition $A_0 \cup \cdots \cup A_{k-1} = \mathbb{N}$ such that the $\overline{A}$'s are hyperimmune.

If suffices to show that every computable RT^2_ℓ-instance has a homogeneous set H such that $\overline{A}_i$ is H-hyperimmune for at least two i's.

COHESIVENESS

Definition

Given a sequence of sets $R_0, R_1, \ldots$, an infinite set C is $\vec{R}$-cohesive if $C \subseteq^* R_i$ or $C \subseteq^* \overline{R_i}$ for each $i \in \mathbb{N}$.

Definition (Cohesiveness)

Every countable sequence of sets $\vec{R}$ admits an $\vec{R}$-cohesive set.

COHESIVENESS AND RT^2_ℓ

- Fix computable instance $f : [\mathbb{N}]^2 \to \ell$ of RT^2_ℓ.
- Define $R_{x,i} = \{y : f(x,y) = i\}$.
- Take an $\vec{R}$-cohesive set C.
- Let $B_i = \{x \in C : \lim_{y \in C} f(x,y) = i\}$

Any infinite subset of one of the B's computes an infinite f-homogeneous set.

RT^2_ℓ AND HYPERIMMUNITY

We need to prove hyperimmunity preservation results for

- Cohesiveness
- Non-effective RT^1_ℓ

PRESERVATION OF HYPERIMMUNITY

Definition

A Π^1_2 statement P admits preservation of hyperimmunity if for each set Z, each sequence of Z-hyperimmune sets $A_0, A_1, \ldots,$ and each P-instance $X \leq_T Z$, there is a solution Y to X such that the A's are $Y \oplus Z$-hyperimmune.

Preservation of hyperimmunity $\neq$ hyperimmune-free solutions

Theorem (Jockusch & Stephan)

If $R_0, R_1, \dots$ are the primitive recursive sets then every $\vec{R}$-cohesive is hyperimmune.

Theorem (P.)

COH *admits preservation of hyperimmunity.*

MATHIAS FORCING

$$(F, X)$$

Initial segment Reservoir

F is finite, X is infinite and $max(F) < min(X)$.

MATHIAS FORCING

A condition (E, Y) extends (F, X) if

(a) $F \subseteq E$

(b) $Y \subseteq X$

(c) $E \setminus F \subseteq X$

A set G satisfies (F, X) if $F \subseteq G$ and $G \setminus F \subseteq X$.

COH ADMITS PRESERVATION OF HYPERIMMUNITY

- Fix a Z and a sequence of Z-hyperimmune sets $A_0, A_1, \ldots$
- Fix a Z-computable sequence $R_0, R_1, \ldots$

We build an $\vec{R}$-cohesive set with Mathias conditions (F, X) where the A's are $X \oplus Z$-hyperimmune.

COH admits preservation of hyperimmunity

Lemma

For every condition c and every pair of indices e, i, there is an extension d of c which forces $\Phi_e^{G \oplus Z}$ not to dominate p_{A_i}.

Proof (Part I).

- Fix $c = (F, X)$.
- Define $f(x) = \begin{cases} \Phi_e^{(F \cup E) \oplus Z}(x) & \text{for some } E \subseteq X \\ \uparrow & \text{otherwise} \end{cases}$
- f is partial $X \oplus Z$-computable.

□

COH ADMITS PRESERVATION OF HYPERIMMUNITY

Lemma

For every condition c and every pair of indices e, i, there is an extension d of c which forces $\Phi_e^{G \oplus Z}$ not to dominate p_{A_i}.

Proof (Part II).

- If f is partial, then c forces $\Phi_e^{G \oplus Z}$ to be partial.
- If f is total, then $f(x) \leq p_{A_i}(x)$ for some x.
  - Let E be such that $f(x) = \Phi_e^{(F \cup E) \oplus Z}(x)$
  - $(F \cup E, X \setminus [0, max(E)])$ forces $f(x) \leq p_{A_i}(x)$

$\square$

NON-EFFECTIVE RT^1_ℓ

Lemma

Δ^0_2-RT^1_ℓ *does not admit preservation of hyperimmunity.*

Proof.

Take $C_0 \cup \cdots \cup C_{\ell-1} = \mathbb{N}$ be hyperimmune sets.
If $H \subseteq C_i$, then p_H dominates p_{C_i}, so C_i is not H-hyperimmune. $\square$

Definition
Given two integers $u, \ell \geq 1$, we let $\pi(u, \ell)$ denote the unique $a \geq 1$ such that $u = a \cdot \ell - b$ for some $b \in [0, \ell)$.

If you have u pigeons in ℓ pigeonholes, one of the holes has at least $\pi(u, \ell)$ pigeons.

PRESERVATION OF HYPERIMMUNITY

Theorem (P.)

Fix some $k \geq 1$ and $\ell \geq 2$ and k hyperimmune sets $A_0, \dots, A_{k-1}$. For every ℓ-partition $B_0 \cup \dots \cup B_{\ell-1} = \omega$, there exists an infinite subset H of some B_i such that $\pi(k, \ell)$ sets among the A's are H-hyperimmune.

Build a set G by Mathias forcing, and let $H = G \cap B_i$ for some $i < \ell$.

NON-EFFECTIVE RT^1_ℓ

Lemma

For every condition c and every pair of indices e, i, there is an extension d of c which forces $\Phi_e^{(G \cap B_j) \oplus Z}$ not to dominate p_{A_i} for some $j < \ell$.

HOW MANY APPLICATIONS?

Theorem

For every $k > \ell \geq 2$, $\mathsf{RT}^2_k \not\leq_c \mathsf{RT}^2_\ell$.

Proof (Part I).

- Define a Δ^0_2 partition $A_0 \cup \cdots \cup A_{k-1} = \mathbb{N}$ such that the $\overline{A}$'s are hyperimmune.
- Consider its Δ^0_2 approximation function as a computable instance of RT^2_k.

$\square$

HOW MANY APPLICATIONS?

Theorem

For every $k > \ell \geq 2$, $\mathsf{RT}^2_k \not\leq_c \mathsf{RT}^2_\ell$.

Proof (Part II).

- Fix computable instance $f : [\mathbb{N}]^2 \to \ell$ of RT^2_ℓ.
- Construct an $\vec{R}$-cohesive set C such that the $\overline{A}$'s are hyperimmune relative to C.
- Let $B_i = \{x \in C : \lim_{y \in C} f(x,y) = i\}$
- Take an infinite subset H of some B_i such that $\pi(k,\ell)$ among the A's are $H \oplus C$-hyperimmune.

$\square$

HOW MANY APPLICATIONS?

Theorem (P.)

For every $k > \ell \geq 2$, $\mathsf{RT}^n_k \not\leq_c \mathsf{RT}^n_\ell$.

Proof.

By induction over $k \geq 2$ using prehomogeneous sets. $\square$

SUMMARY FOR A FIXED n

$$\begin{array}{cc} & \vdots \\ & \downarrow \\ & \mathsf{RT}^n_4 \\ & \downarrow \\ & \mathsf{RT}^n_3 \\ & \downarrow \\ \mathsf{RT}^n_k, k \geq 2 & \mathsf{RT}^n_2 \end{array}$$

Over RCA_0 Over $\leq_c$

COUNTING APPLICATIONS

Question

How many applications needed to prove that $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \to \mathsf{RT}^2_5$*?*

Take a Δ^0_2 5-partition $A_0 \cup \cdots \cup A_4 = \mathbb{N}$ whose complements are hyperimmune.

# of apps of RT^2_2	# of i's such that $\overline{A}_i$ is hyperimmune
0	5
1	$\pi(5,2) = 3$
2	$\pi(3,2) = 2$
3	$\pi(2,2) = 1$

RAMSEY'S THEOREM

Over n-tuples

$$\mathsf{RT}^n_k$$

Using k colors

RAMSEY'S THEOREM

Over n-tuples

$$\mathsf{RT}^{n}_{k,r}$$

Allows r colors

Using k colors

THIN SET THEOREM

$$\mathsf{TS}^{n}_{k} \qquad\qquad \mathsf{RT}^{n}_{k,k-1}$$

Allowing more colors

Theorem (Wang, 2014)

Fix some n and some sufficiently large k's. For every instance f of TS^n_k and every non-computable set C, there is an infinite solution to f which does not compute C.

Corollary

For every n and sufficiently large k, TS^n_k does not imply RT^3_2 over RCA_0.

ALLOWING MORE COLORS

Theorem (Dorais, Dzhafarov, Hirst, Mileti, Shafer, 2015)
$\mathsf{RCA}_0 \vdash \mathsf{TS}^{ns+1}_{k^s} \to \mathsf{TS}^{n+1}_{k}$

Theorem (Dorais, Dzhafarov, Hirst, Mileti, Shafer, 2015)
$\mathsf{RCA}_0 \vdash \mathsf{TS}^{n+2}_{2^n} \to \mathsf{TS}^{3}_{2}$

ALLOWING MORE COLORS

Tuples	Strong avoidance	Computes $\emptyset'$
TS^1_k	$k \geq 2$	never
TS^2_k	$k \geq 3$	$k = 2$
TS^3_k	$k \geq 7$	$k \leq 4$

Does any of TS^3_5 or TS^3_6 admit strong cone avoidance?

ALLOWING MORE COLORS

Theorem (P.)

For every $k \geq 2$,

- TS^2_{k+1} *admits preservation of k hyperimmunities.*
- TS^2_k *does not admit preservation of k hyperimmunities.*

Corollary (P.)

For every $k \geq 2$, TS^2_{k+1} does not imply TS^2_k over RCA_0.

ALLOWING MORE COLORS

Fix some $\ell \geq 2$.

Theorem (P.)

For every n and sufficiently large k's,
TS^n_k *admits preservation of* ℓ *hyperimmunities.*

Corollary (P.)

For every n and sufficiently large k's,
TS^n_k *does not imply* TS^2_ℓ *over* RCA_0.

SUMMARY FOR $n = 2$

$$\mathsf{RT}^2_2$$
$$\downarrow$$
$$\mathsf{TS}^2_3$$
$$\downarrow$$
$$\mathsf{TS}^2_4$$
$$\vdots$$
$$\downarrow$$

Over RCA_0

CONCLUSION

- Computable reducibility gives a more fine-grained analysis than reverse mathematics.
- Ramsey's theorem is not robust for computable reducibility.
- Changing the number of allowed colors has a great impact on the strength of Ramsey's theorem.

REFERENCES

Peter A. Cholak, Carl G. Jockusch, and Theodore A. Slaman.
On the strength of Ramsey's theorem for pairs.
Journal of Symbolic Logic, pages 1–55, 2001.

François G Dorais, Damir D Dzhafarov, Jeffry L Hirst, Joseph R Mileti, and Paul Shafer.
On uniform relationships between combinatorial problems.
arXiv preprint arXiv:1212.0157, 2012.

Carl G Jockusch.
Ramsey's theorem and recursion theory.
Journal of Symbolic Logic, 37(2):268–280, 1972.

Ludovic Patey.
The weakness of being cohesive, thin or free in reverse mathematics.
Submitted, 2015.

Wei Wang.
Some logically weak ramseyan theorems.
Advances in Mathematics, 261:1–25, 2014.

Questions

Thank you for listening!