DEGREES BOUNDING PRINCIPLES AND UNIVERSAL INSTANCES IN REVERSE MATHEMATICS

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ABSTRACT. A Turing degree $\mathbf{d}$ *bounds* a principle P of reverse mathematics if every computable instance of P has a $\mathbf{d}$-computable solution. P admits a *universal instance* if there exists a computable instance such that every solution bounds P. We prove that the stable version of the ascending descending sequence principle (SADS) as well as the stable version of the thin set theorem for pairs ($\mathsf{STS}(2)$) do not admit a bound of low$_2$ degree. Therefore no principle between Ramsey's theorem for pairs (RT^2_2) and SADS or $\mathsf{STS}(2)$ admit a universal instance. We construct a low$_2$ degree bounding the Erdős Moser theorem (EM), thereby showing that the previous argument does not hold for EM. Finally, we prove that the only Δ^0_2 degree bounding a stable version of the rainbow Ramsey theorem for pairs (SRRT^2_2) is $\mathbf{0}'$. Hence no principle between the stable Ramsey theorem for pairs (SRT^2_2) and SRRT^2_2 admit a universal instance. In particular the stable version of the Erdős-Moser theorem does not admit one. It remains unknown whether EM admits a universal instance.

1. INTRODUCTION

Reverse mathematics is a program whose goal is to classify theorems according to their computational strength, within the framework of subsystems of second-order arithmetic. Proofs are done relatively to a very weak system (RCA_0) meant to capture *computational mathematics*. RCA_0 is composed of basic Peano axioms, Δ^0_1 comprehension and Σ^0_1 induction schemes. See [12] for a good introductory book. Most of statements in reverse mathematics are of the form

$$\forall X(\Phi(X) \to \exists Y \Psi(X,Y))$$

where Φ and Ψ are arithmetic formulas.

A set X such that $\Phi(X)$ holds is called an *instance* of P and a set Y such that $\Psi(X,Y)$ holds is a *solution* to X. We can see relations between two instances X_1, X_2 of a statement P as a mass problem consisting of computing a solution to X_1 given any solution to X_2.

Definition 1.1 Given a statement P, a degree $\mathbf{d}$ is P-*bounding* ($\mathbf{d} \gg_{\mathsf{P}} \emptyset$) if every computable instance X of P has a $\mathbf{d}$-computable solution. A statement P admits a *universal instance* if it has a computable instance X such that every solution to X bounds P.

The notation $\mathbf{d} \gg \emptyset$ historically means that the degree $\mathbf{d}$ is PA and therefore is equivalent to $\mathbf{d} \gg_{\mathsf{WKL}_0} \emptyset$ where WKL_0 is weak König's lemma principle, i.e., König's lemma restricted to subtrees of $2^{<\omega}$. It is well-known that WKL_0 admits a universal instance – e.g. take the Π^0_1 class of completions of Peano arithmetics –. A few principles have been proven to admit universal instances – WKL_0 [22], König's lemma (KL) [12], the

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Ramsey-type weak weak König's lemma (RWWKL) [1], the finite intersection property (FIP) [9], the omitting partial type theorem (OPT) [15], or even the rainbow Ramsey theorem for pairs (RRT^2_2) [21] – but most of principles do not admit one. An important notion for proving such a result is computable reducibility.

Definition 1.2 Fix two statements P and Q. We say that P is *computably reducible* to Q (written $\mathsf{P} \leq_c \mathsf{Q}$) if for every instance X of P there is an X-computable instance Y of Q such that each solution to Y computes relative to X a solution to X. P and Q are *computably equivalent* if $\mathsf{P} \leq_c \mathsf{Q}$ and $\mathsf{Q} \leq_c \mathsf{P}$.

Mileti proved in [20] that the stable Ramsey theorem for pairs (SRT^2_2) admits no bound of low$_2$ degree. Therefore every statement P having an ω-model with only low$_2$ sets, and such that $\mathsf{SRT}^2_2 \leq_c \mathsf{P}$, admits no universal instance. In particular none of Ramsey's theorem for pairs (RT^2_2), SRT^2_2 and the Ramsey-type weak König's lemma relative to $\emptyset'$ ($\mathsf{RWKL}[\emptyset']$) admit a universal instance. Independently, Hirschfeldt & Shore proved in [14] that the stable ascending descending sequence principle (SADS) admits no bound of low degree. Hence neither SADS nor the stable chain antichain principle (SCAC) admit a universal instance.

We generalize both results by proving that SADS does not admit a bound of low$_2$ degree, proving therefore that if a statement P has an ω-model with only low$_2$ sets and $\mathsf{SADS} \leq_c \mathsf{P}$ then P admits no universal instance. We also extend the result to statements to which the stable thin set theorem for pairs ($\mathsf{STS}(2)$) computably reduces. Hence we deduce that none of the ascending descending sequence principle (ADS), the chain antichain principle (CAC), the thin set theorem for pairs ($\mathsf{TS}(2)$), the free set theorem for pairs ($\mathsf{FS}(2)$) and their stable versions admit a universal instance.

We generalize the result to arbitrary tuples and prove that none of RT^n_2, $\mathsf{FS}(n)$, $\mathsf{TS}(n)$ and their stable versions admit a universal instance for $n \geq 2$. The question remains open for the rainbow Ramsey theorem for n-tuples (RRT^n_2) with $n \geq 3$. We construct a low$_2$ degree bounding the Erdős Moser theorem (EM), thereby showing that the previous argument does not hold for EM.

Mileti proved in [20] that the only Δ^0_2 degree bounding SRT^2_2 is $\mathbf{0}'$. Using the fact that every Δ^0_2 set has an infinite incomplete Δ^0_2 subset in either it or its complement [13], we obtain another proof that SRT^2_2 admits no universal instance. We extend this result by proving that the only Δ^0_2 degree bounding a stable version of the rainbow Ramsey theorem for pairs (SRRT^2_2) is $\mathbf{0}'$. Hence none of the statements P satisfying $\mathsf{SRRT}^2_2 \leq_c \mathsf{P} \leq_c \mathsf{SRT}^2_2$ admit a universal instance. In particular we deduce that neither SRRT^2_2 nor the stable version of the Erdős-Moser theorem (SEM) admits a universal instance.

1.1. **Notation.** *Formulas*. The notation $(\forall^\infty s)\varphi(s)$ means that $\varphi(s)$ holds for all but finitely many s, i.e., is translated to $(\exists s_0)(\forall s \geq s_0)\varphi(s)$. Given two sets X and Y, we denote by $X \subseteq^* Y$ the statement $(\forall^\infty s \in X)[s \in Y]$. Accordingly, $X =^* Y$ means that both $X \subseteq^* Y$ and $Y \subseteq^* X$ hold, i.e., X and Y differ by finitely many elements.

Turing functional and lowness. We fix an effective enumeration of all Turing functionals $\Phi_0, \Phi_1, \dots$ We denote by $\Phi_{e,s}$ the partial approximation of the Turing functional Φ_e at stage s. Given a set X, we denote by X' the jump of X and by $X^{(n)}$ the nth jump of X. A set X is *low$_n$ over Y* if $(X \oplus Y)^{(n)} \leq Y^{(n)}$. A set is *low$_n$* if it is low$_n$ over $\emptyset$. A *low$_n$-ness* index of a set X low$_n$ over Y is a Turing index e such that $\Phi_e^{Y^{(n)}} = (X \oplus Y)^{(n)}$.

Mathias forcing. Given two sets E and F, we denote by $E < F$ the formula $(\forall x \in E)(\forall y \in F) x < y$. A *Mathias condition* is a pair (F, X) where F is a finite set, X is an infinite set and $F < X$. A condition $(\tilde{F}, \tilde{X})$ *extends* (F, X) (written $(\tilde{F}, \tilde{X}) \leq (F, X)$) if $F \subseteq \tilde{F}$, $\tilde{X} \subseteq X$ and $\tilde{F} \setminus F \subset X$. A set G *satisfies* a Mathias condition (F, X) if $F \subset G$ and $G \setminus F \subseteq X$.

2. Degrees bounding cohesiveness

A standard proof of Ramsey's theorem for pairs consists of reducing an arbitrary coloring of pairs into a *stable* one using the cohesiveness principle. The understanding of the links between cohesiveness and stability is a very active subject of research in reverse mathematics [4, 13, 5].

Definition 2.1 (Cohesiveness) An infinite set C is $\vec{R}$-cohesive for a sequence of sets $R_0, R_1, \dots$ if for each $i \in \omega$, $C \subseteq^* R_i$ or $C \subseteq^* \overline{R_i}$. A set C is *cohesive* (resp. *r-cohesive*) if it is $\vec{R}$-cohesive where $\vec{R}$ is an enumeration of all c.e. (resp. computable) sets. COH is the statement "Every uniform sequence of sets $\vec{R}$ has an $\vec{R}$-cohesive set."

Jockusch et al. proved in [16] the existence of a low$_2$ cohesive set. Degrees bounding COH are quite well understood and admit a simple characterization:

Theorem 2.2 (Jockusch & Stephan [16]) Fix an $n \in \omega$.

1. For every set C such that $C' \gg \emptyset'$, $C \gg_{\mathsf{COH}} \emptyset$.
2. There exists a uniformly $\emptyset^{(n)}$-computable sequence of sets $\vec{R}$ such that for every $\vec{R}$-cohesive set C, $(C \oplus \emptyset^{(n)})' \gg \emptyset^{(n+1)}$.

In particular, taking a set $P \gg \emptyset'$ low over $\emptyset'$ and a set C such that $C' =_T P$ whose existence is ensured by Friedberg's jump inversion theorem, we obtain a low$_2$ degree bounding COH. The canonical $\emptyset^{(n)}$-computable sequence of sets $\vec{R}$ whose existence is claimed in clause 2 of Theorem 2.2 is

$$R_e = \{s : \Phi_{e,s}^{\emptyset_s^{(n+1)}}(e) \downarrow = 1\}$$

Every $\vec{R}$-cohesive set C computes a function $f(\cdot,\cdot)$ such that $\lim_{s \in C} f(e,s)$ exists for each $e \in \omega$ and $\lim_{s \in C} f(e,s) = \Phi_e^{\emptyset^{(n+1)}}(e)$ for each Turing index e such that $\Phi_e^{\emptyset^{(n+1)}}(e) \downarrow$. By a relativized version of Schoenfield's limit lemma, $(C \oplus \emptyset^{(n)})'$ computes the function $\tilde{f}(x) = \lim_{s \in C} f(x,s)$ and is therefore of PA degree relative to $\emptyset^{(n+1)}$.

Corollary 2.3 COH admits a universal instance.

Proof. The uniformly computable sequence of sets $\vec{R}$ such that the jump of every $\vec{R}$-cohesive set is of PA degree relative to $\emptyset'$ is a universal instance by the previous theorem. □

Wang proved in [26] that for every set $P \gg \emptyset''$ and every uniformly $\emptyset'$-computable sequence of sets $\vec{R}$, there exists an $\vec{R}$-cohesive set C such that $C'' \leq_T C \oplus \emptyset'' \leq_T P$. Cholak et al. used in [4] the existence of a low subuniform degree to deduce the existence, for every set $P \gg \emptyset'$, of an r-cohesive set C such that $C' \leq_T P$. We can apply a similar reasoning for $\emptyset'$-computable sets, using the fact that degrees bounding COH are somehow subuniform degrees for Δ_2^0 approximations.

Theorem 2.4 For every set $P \gg \emptyset''$, there exists an $\vec{R}$-cohesive set C such that $C'' \leq_T C \oplus \emptyset'' \leq_T P$, where $\vec{R}$ is the (non-uniformly computable) sequence of all $\emptyset'$-computable sets.

Proof. Let $\vec{U}$ be the uniformly computable sequence of sets defined by

$$U_{e,x} = \{s : \Phi_{e,s}^{\emptyset'_s}(x) = 1\}$$

Fix a low$_2$ $\vec{U}$-cohesive set C_0 and its C_0-computable bijection $f : \omega \to C_0$. Every set $P \gg \emptyset''$, $P \gg C_0''$. Consider the uniformly C_0'-computable sequence of sets

$$V_e = \{x : \lim_s \Phi_{e,s}^{\emptyset'_{f(s)}}(x) = 1\}$$

The sequence $\vec{V}$ contains every $\emptyset'$-computable set. In particular, every $\vec{V}$-cohesive set is $\vec{R}$-cohesive. By a relativization of Wang's result, there exists an $\vec{V}$-cohesive set C such that $(C \oplus C_0)'' \leq_T C \oplus C_0'' =_T C \oplus \emptyset'' \leq_T P$. □

The proof of the previous theorem shows that an application of COH followed by an application of $\mathsf{COH}[\emptyset']$ are enough to obtain a set of degree bounding $\mathsf{COH}[\emptyset']$. The following question remains open:

Question 2.5 Does $\mathsf{COH}[\emptyset']$ admit a universal instance?

3. Degrees bounding the atomic model theorem

The atomic model theorem is a statement of model theory admitting a simple, purely computability-theoretic characterization over ω-models. This statement happens to have a weak computational content and is therefore a consequence of many other principles in reverse mathematics. For those reasons, the atomic model theorem is a good candidate for factorizing proofs of properties which are closed upward by the consequence relation.

Definition 3.1 (Atomic model theorem) A formula $\varphi(x_1, \dots, x_n)$ of T is an *atom* of a theory T if for each formula $\psi(x_1, \dots, x_n)$, one of $T \vdash \varphi \to \psi$ and $T \vdash \varphi \to \neg\psi$ holds, but not both. A theory T is *atomic* if, for every formula $\psi(x_1, \dots, x_n)$ consistent with T, there exists an atom $\varphi(x_1, \dots, x_n)$ of T extending it, i.e., one such that $T \vdash \varphi \to \psi$. A model $\mathscr{A}$ of T is *atomic* if every n-tuple from $\mathscr{A}$ satisfies an atom of T. AMT is the statement "Every complete atomic theory has an atomic model".

AMT has been introduced as a principle by Hirschfeldt et al. in [15]. They proved that WKL_0 and AMT are incomparable on ω-models, proved over RCA_0 that AMT is strictly weaker than SADS. The author proved in [23] that $\mathsf{STS}(2)$ implies AMT over RCA_0. In this section we use the fact that AMT is not bounded by any Δ^0_2 low$_2$ degree to deduce that none of AMT, SADS and SCAC admits a universal instance. The principle AMT has been proven in [15, 6] to be computably equivalent to the following principle:

Definition 3.2 (Escape property) For every Δ^0_2 function f, there exists a function g such that $f(x) \leq g(x)$ for infinitely many x.

This equivalence does not hold over RCA_0 as, unlike AMT, the escape property implies $\mathsf{I}\Sigma^0_2$ over $\mathsf{B}\Sigma^0_2$ [15]. Using this characterization, we can easily deduce the two following theorems:

Theorem 3.3 (Hirschfeldt et al. [15]) There is no low$_2$ Δ^0_2 degree bounding AMT.

Theorem 3.4 No principle P having an ω-model with only low sets and such that $\mathsf{AMT} \leq_{\mathrm{c}} \mathsf{P}$ admits a universal instance.

Theorem 3.3 and Theorem 3.4 can be easily proven using the following characterization of Δ^0_2 low$_2$ sets in terms of domination:

Lemma 3.5 (Martin, [19]) A set $A \leq_T \emptyset'$ is low$_2$ iff there exists an $f \leq_T \emptyset'$ dominating every A-computable function.

Proof. A set A is low$_2$ iff $\emptyset'$ is high relative to A. We conclude the lemma from the observation that a set X is high relative to a set $A \leq_T \emptyset'$ iff it computes a function dominating every A-computable function. $\square$

Remark. As explained Conidis in [6], Theorem 3.3 cannot be extended to every low$_2$ sets: Soare [6] constructed a low$_2$ set bounding the escape property using a forcing argument. So there exists a low$_2$ degree bounding AMT.

Proof of Theorem 3.4. Suppose for the sake of contradiction that P has a universal instance U and an ω-model $\mathscr{M}$ with only low sets. As U is computable, $U \in \mathscr{M}$. Let $X \in \mathscr{M}$ be a (low) solution to U. In particular, X is low$_2$ and Δ^0_2, so by Lemma 3.5 and the computable equivalence of AMT and the escape property, there exists a computable instance Y of AMT such that X does not compute a solution to Y. As $\mathsf{AMT} \leq_{\mathrm{c}} \mathsf{P}$, there exists a Y-computable (hence computable) instance Z of P such that every solution to Z computes a solution to Y. Thus X does not compute a solution to Z, contradicting universality of U. $\square$

Hirschfeldt et al. proved in [14] the existence of an ω-model of SADS and SCAC with only low sets. Therefore we obtain another proof that neither SADS nor SCAC admits a universal instance. The result was first proven in [14] using an ad-hoc notion of reducibility.

Corollary 3.6 None of AMT, SADS and SCAC admit a universal instance.

The previous argument cannot directly be applied to SRT^2_2, SEM or $\mathsf{STS}(2)$ as none of those principles admit an ω-model with only low sets [10, 17, 23]. However Lemma 3.4 can be extended to principles such that every computable instance has a Δ^0_2 low$_2$ solution. It is currently unknown whether every Δ^0_2 set admits a Δ^0_2 low$_2$ infinite subset in either it or its complement. A positive answer would lead to a proof that SRT^2_2, SEM and $\mathsf{STS}(2)$ have no universal instance, and more importantly, would provide an ω-model of SRT^2_2 that is not a model of $\mathsf{DNR}[\emptyset']$ as explained in [13]. We shall see later that none of SRT^2_2, SEM and $\mathsf{STS}(2)$ admits a universal instance.

4. Degrees bounding $\mathsf{STS}(2)$ and SADS

Mileti originally proved in [20] that no principle P having an ω-model with only low$_2$ sets and satisfying $\mathsf{SRT}^2_2 \leq_{\mathrm{c}} \mathsf{P}$ admits a universal instance, and deduced that none of

SRT^2_2 and RT^2_2 admit one. In this section, we reapply his argument to much weaker statements and derive non-universality results to a large range of principles in reverse mathematics. Thin set theorem and ascending descending sequence are example of statements weak enough to be a consequence of many others, and surprisingly strong enough to diagonalize against low$_2$ sets.

Definition 4.1 (Thin set) Let $k \in \omega$ and $f : [\omega]^k \to \omega$. A set A is *thin for* f if $f([A]^k) \neq \omega$, that is, if the set A "avoids" at least one color. $\mathsf{TS(k)}$ is the statement "every function $f : [\omega]^k \to \omega$ has an infinite set thin for f". A function $f : [\omega]^k \to \omega$ is *stable* if $\forall \sigma \in [\omega]^{k-1}$, $\lim_s f(\sigma, s)$ exists. $\mathsf{STS(k)}$ is the restriction of $\mathsf{TS(k)}$ to stable functions.

Cholak et al. studied extensively thin set principle in [3]. Some of the results where already stated by Friedman without giving a proof, notably there exists an ω-model of WKL_0 which is not a model of $\mathsf{TS}(2)$, and the arithmetical comprehension axiom (ACA_0) does not imply $(\forall k)\mathsf{TS(k)}$ over RCA_0. Wang showed in [28] that $(\forall k)\mathsf{TS(k)}$ does not imply ACA_0 on ω-models. Rice [24] proved that $\mathsf{STS}(2)$ implies DNR over RCA_0. The author proved in [23] that $\mathsf{RCA}_0 \vdash \mathsf{TS}(2) \to \mathsf{RRT}^2_2$.

Definition 4.2 (Ascending descending sequence) ADS is the statement "Every infinite linear order admits an infinite ascending or descending sequence". SADS is the restriction of ADS to order types $\omega + \omega^*$.

Tennenbaum [25] constructed a computable linear order of order type $\omega + \omega^*$ with no computable ascending or descending sequence. Therefore SADS does not hold over RCA_0. Hirschfeldt & Shore [14] studied ADS within the framework of reverse mathematics, proving that ADS implies both COH and $\mathsf{B}\Sigma^0_2$ over RCA_0 and that SADS implies AMT over RCA_0. They constructed an ω-model of ADS that is not a model of DNR, and an ω-model of $\mathsf{COH} + \mathsf{WKL}_0$ that is not a model of SADS.

The study of degrees bounding a statement and the existence of a universal instance are closely related. As does Mileti in [20], we deduce two kind of theorems by the application of his proof technique.

Theorem 4.3 There exists no low$_2$ degree bounding any of $\mathsf{STS}(2)$ or SADS.

Theorem 4.4 No principle P having an ω-model with only low$_2$ sets and such that any of $\mathsf{STS}(2)$, SADS is computably reducible to P admits a universal instance.

The proof of the two theorems is split into three lemmas. Lemma 4.7 provides a general way of obtaining bounding and universality results, assuming the ability of a principle to diagonalize against a particular set. Lemma 4.8 and Lemma 4.9 state the desired diagonalization for respectively $\mathsf{STS}(2)$ and SADS.

Corollary 4.5 None of the following principles admits a universal instance: RT^2_2, $\mathsf{RWKL}[\emptyset']$, $\mathsf{FS}(2)$, $\mathsf{TS}(2)$, CAC, ADS and their stable versions.

Proof. Each of the above mentioned principles is a consequence of RT^2_2 over RCA_0 and computably implies either SADS or $\mathsf{STS}(2)$. See [11] for $\mathsf{RWKL}[\emptyset']$, [3] for $\mathsf{FS}(2)$ and $\mathsf{TS}(2)$, and [14] for CAC and ADS. By Theorem 3.1 of [4], there exists an ω-model of RT^2_2 having only low$_2$ sets. The result now follows from Theorem 4.4. $\square$

In order to prove Theorem 4.3 and Theorem 4.4, we need the following theorem proven by Mileti. It simply consists of applying a relativized version of the low basis theorem to a Π^0_1 class of completions of the enumeration of all partial computable sets.

Theorem 4.6 (Mileti, Corollary 5.4.5 of [20]) For every set X, there exists $f : \omega^2 \to \{0,1\}$ low over X such that for every X-computable set Z, there exists an $e \in \omega$ with $Z = \{a \in \omega : f(e,a) = 1\}$.

Lemma 4.7 Fix an $n \in \omega$ and two principles P and Q such that $\mathsf{P} \leq_c \mathsf{Q}$. Suppose that for any $f : \omega^2 \to \{0,1\}$ satisfying $f'' \leq_T \emptyset^{(n+2)}$, there exists a computable instance I of P such that for each $e \in \omega$, if $\{a \in \omega : f(e,a) = 1\}$ is infinite then it is not a solution to I. Then the following holds:

(i) For any degree $\mathbf{d}$ low$_2$ over $\emptyset^{(n)}$ there is a computable instance U of P such that $\mathbf{d}$ does not bound a solution to U.
(ii) There is no degree low$_2$ over $\emptyset^{(n)}$ bounding P.
(iii) If every computable instance I of Q has a solution low$_2$ over $\emptyset^{(n)}$, then Q has no universal instance.

Proof.

(i) Consider any set X of degree low$_2$ over $\emptyset^{(n)}$. By Theorem 4.6, there exists a function $f : \omega^2 \to \{0,1\}$ low over X, hence low$_2$ over $\emptyset^{(n)}$, such that any X-computable set Z is of the form $\{a \in \omega : f(e,a) = 1\}$ for some $e \in \omega$. Take a computable instance I of P having no solution of the form $\{a \in \omega : f(e,a) = 1\}$ for any $e \in \omega$. Then X does not compute a solution to I.
(ii) Immediate from (i).
(iii) Take any computable instance U of Q. By assumption, U has a solution X low$_2$ over $\emptyset^{(n)}$. By (i), there exists an instance I of P such that X does not compute a solution to I. As $\mathsf{P} \leq_c \mathsf{Q}$, there exists an I-computable (hence computable) instance J of Q such that any solution to J computes a solution to I. Then X does not compute a solution to J, hence U is not a universal instance.

□

We will prove the following lemmas which, together with Lemma 4.7, are sufficient to deduce Theorem 4.3 and Theorem 4.4.

Lemma 4.8 Fix a set X. Suppose $f : \omega^2 \to \{0,1\}$ satisfies $f'' \leq_T X''$. There exists an X-computable stable coloring $g : [\omega] \to \omega$ such that for all $e \in \omega$, if $\{a \in \omega : f(e,a) = 1\}$ is infinite then it is not thin for g.

Lemma 4.9 Fix a set X. Suppose $f : \omega^2 \to \{0,1\}$ satisfies $f'' \leq_T X''$. There exists a stable X-computable linear order L such that for all $e \in \omega$, if $\{a \in \omega : f(e,a) = 1\}$ is infinite then it is neither an ascending nor a descending sequence in L.

Before proving the two remaining lemmas, we relativize the results to colorings over arbitrary tuples.

Theorem 4.10 For any n, there exists no degree low$_2$ over $\emptyset^{(n)}$ bounding $\mathsf{STS}(\mathsf{n}+2)$.

Proof. Apply Lemma 4.8 relativized to $X = \emptyset^{(n)}$ together with Lemma 4.7. Simply notice that if $f : [\omega]^n \to \omega$ is a $\emptyset'$-computable coloring, the computable coloring $g : [\omega]^{n+1} \to \omega$ obtained by an application of Schoenfield's limit lemma is such that every infinite set thin for g is thin for f. □

Theorem 4.11 For any n, no principle P having an ω-model with only low$_2$ over $\emptyset^{(n)}$ sets and such that $\mathsf{STS(n+2)} \leq_c \mathsf{P}$ admits a universal instance.

Proof. Same reasoning as Theorem 4.4 using the notice in the proof of Theorem 4.10. □

Theorem 4.12 For any n, none of RT^{n+2}_2, $\mathsf{RWKL}[\emptyset^{(n+1)}]$, $\mathsf{FS(n+2)}$, $\mathsf{TS(n+2)}$ and their stable versions admits a universal instance.

Proof. Fix an $n \in \omega$. Each of the above cited principles P satisfies $\mathsf{STS(n+2)} \leq_c \mathsf{P}$ and is a consequence of RT^{n+2}_2 over ω-models. Cholak et al. [4] proved the existence of an ω-model of RT^{n+2}_2 having only low$_2$ over $\emptyset^{(n)}$ sets. Apply Theorem 4.11. □

We now turn to the proofs of Lemma 4.8, and Lemma 4.9.

Proof of Lemma 4.8. We prove it in the case when $X = \emptyset$. The general case follows by a straightforward relativization. For each $e \in \omega$, let $Z_e = \{a \in \omega : f(e,a) = 1\}$. The proof is very similar to [20, Theorem 5.4.2.]. We build a $\emptyset'$-computable function $c : \omega \to \omega$ such that for all $e \in \omega$, if Z_e is infinite then it is not thin for c. Given such a function c, we can then apply Schoenfield's limit lemma to obtain a stable computable function $h : [\omega]^2 \to \omega$ such that for each $x \in \omega$, $\lim_s h(x,s) = c(x)$. Every set thin for h is thin for c, and therefore for all $e \in \omega$, if Z_e is infinite then it is not thin for h.

Suppose by Kleene's fixpoint theorem that we are given a Turing index d of the function c as computed relative to $\emptyset'$. The construction is done by a finite injury priority argument satisfying the following requirements for each $e, i \in \omega$:

$$\mathscr{R}_{e,i} : Z_e \text{ is finite or } (\exists a)[f(e,a) = 1 \text{ and } \Phi^{\emptyset'}_d(a) = i]$$

The requirements are ordered in a standard way, that is, following the pairing of the indexes. Notice that each of these requirement is Σ^f_2, and furthermore we can effectively find an index for each as such. Therefore, for each e and $i \in \omega$, we can effectively find an integer $m_{e,i}$ such that $R_{e,i}$ is satisfied if and only if $m_{e,i} \in f''$. By Schoenfield's limit Lemma relativized to $\emptyset'$ and low$_2$-nes of f, there exists a $\emptyset'$-computable function $g : \omega^2 \to 2$ such that for all m, we have $m \in f'' \leftrightarrow \lim_s g(m,s) = 1$ and $m \notin f'' \leftrightarrow \lim_s g(m,s) = 0$. Notice that for all e and $i \in \omega$, $R_{e,i}$ is satisfied if and only if $\lim_s g(m_{e,i}, s) = 1$.

At stage s, assume we have defined $c(u)$ for every $u < s$. If there exists a least strategy $\mathscr{R}_{e,i}$ (in priority order) with $\langle e,i \rangle < s$ such that $g(m_{e,i}, s) = 0$, set $c(s) = i$. Otherwise set $c(s) = 0$. This ends the construction. We now turn to the verification.

Claim. Every requirement $\mathscr{R}_{e,i}$ is satisfied.

Proof. By induction over ordered pairs $\langle e,i \rangle$ in lexicographic order. Suppose that $R_{e',i'}$ is satisfied for all $\langle e',i' \rangle < \langle e,i \rangle$, but $\mathscr{R}_{e,i}$ is not satisfied. Then there exists a threshold $t \geq \langle e,i \rangle$ such that $g(m_{e',i'}, s) = 1$ for all $\langle e',i' \rangle < \langle e,i \rangle$ and $g(m_{e,i}, s) = 0$ whenever $s \geq t$. By construction, $c(s) = i$ for every $s \geq t$. As Z_e is infinite, there exists an element

$s \in Z_e$ such that $c(s) = i$, so Z_e is not thin for c with witness i and therefore $\mathscr{R}_{e,i}$ is satisfied. Contradiction. $\square$

$\square$

Proof of Lemma 4.9. Again, we prove it in the case when $X = \emptyset$. For each $e \in \omega$, let $Z_e = \{a \in \omega : f(e,a) = 1\}$. The proof is very similar to [20, Theorem 5.4.2.]. We build a Δ^0_2 set U together with a stable computable linear order L such that U is the ω part of L, that is, U is the collection of elements L-below cofinitely many other elements. We furthermore ensure that for each $e \in \omega$, if Z_e is infinite, then it intersects both U and $\overline{U}$. Therefore, if Z_e is infinite, it is neither an ascending, nor a descending sequence in L as otherwise it would be included in either U or $\overline{U}$.

Assume by Kleene's fixpoint theorem that we are given the Turing index d of U as computed relative to $\emptyset'$. The set U is built by a finite injury priority construction with the following requirements for each $e \in \omega$:

- $\mathscr{R}_{2e}$: Z_e is finite or $(\exists a)[f(e,a) = 1$ and $\Phi^{\emptyset'}_d(a) = 1]$
- $\mathscr{R}_{2e+1}$: Z_e is finite or $(\exists a)[f(e,a) = 1$ and $\Phi^{\emptyset'}_d(a) = 0]$

Notice again that each of these requirement is Σ^f_2, and furthermore we can effectively find an index for each as such. Therefore, for each $i \in \omega$, we can effectively find an m_i such that R_i is satisfied if and only if $m_i \in f''$. By two applications of Schoenfield's limit Lemma and low$_2$-ness of f, there exists a computable function $g : \omega^3 \to 2$ such that for all $m \in \omega$, we have $m \in f'' \leftrightarrow \lim_t \lim_s g(m,s,t) = 1$ and $m \notin f'' \leftrightarrow \lim_t \lim_s g(m,s,t) = 0$. Notice that for all $i \in \omega$,

$$R_i \text{ is satisfied } \leftrightarrow \lim_t \lim_s g(m_i, s, t) = 1$$

At stage 0, $U_0 = \emptyset$ and every integer is a *leader* and *follows* itself. We say that $\mathscr{R}_i$ *requires attention for u at stage s* if $i \leq u \leq s$, u is *leader* and $g(m_i, s, u) = 0$. At stage $s+1$, assume we have decided $u <_L v$ or $u >_L v$ for every $u, v < s$. Set $u <_L s$ if $u \in U_s$ and $u >_L s$ if $u \notin U_s$. Initially set $U_{s+1} = U_s$. For each leader $u \leq s$ which has not been claimed at stage $s+1$ and for which some requirement $\mathscr{R}_i$, $i < u$ requires attention, say that the least such $\mathscr{R}_i$ *claims* u and act as follows.

(a) If $i = 2e$ and $u \notin U_s$, then add $[u,s]$ to U_{s+1}, where the interval $[u,s]$ is taken in the usual order on ω and not in $<_L$. Elements of $[u+1,s]$ *follow* u and are no more considered as leaders from now on and at any further stage.
(b) If $i = 2e+1$ and $u \in U_s$, then remove $[u,s]$ from U_{s+1}. Similarly, elements of $[u+1,s]$ are no more leaders and *follow* u.

Then go to the next leader $u \leq s$. This ends the construction. An immediate verification shows that at every stage,

- if u stops being a leader it never becomes again a leader
- if u follows v then $v \leq u$, v is a leader, every w between v and u follows v and thus u will never follow any $w > v$.

So the leader that u follows eventually stabilizes. Moreover, because g is limit-computable, each leader eventually stops increasing the number of followers and therefore there are infinitely many leaders.

Claim. L is a linear order.

Proof. As L is a tournament, it suffices to check there is no 3-cycle. By symmetry, we check only the case where $u <_L s <_L v <_L u$ forms a 3-cycle with s the maximal element in $<_\omega$ order. By construction, this means that $u \in U_s$, $v \notin U_s$. If $u <_\omega v$, then $u \notin U_v$ and so there exists a leader $w \leq_\omega u$ and an even number $i \leq w$ such that $\mathscr{R}_i$ requires attention for w at a stage $t \geq v$. Case (a) of the construction applies and the interval $[w+1, t]$ is included U at least until stage s. As $v \in [w+1, t]$, $v \in U_s$ contradicting our hypothesis. Case $u >_\omega v$ is symmetric. □

Claim. U is Δ^0_2.

Proof. Suppose for the sake of contradiction that there exists a least element u entering U and leaving it infinitely many times. Such a u must be a leader, otherwise it would not be the least one. Let $\mathscr{R}_i$ be the least requirement claiming u infinitely many times. As $\lim_s g(m_i, s, u)$ exists, it will claim u cofinitely many times and therefore u will be in U or in $\overline{U}$ cofinitely many times. Contradiction. □

It immediately follows that L is stable.

Claim. Every requirement $\mathscr{R}_i$ is satisfied.

Proof. By induction over R_i in priority order. Suppose that R_j is satisfied for all $j < i$, but $\mathscr{R}_i$ is not satisfied. Then there exists a threshold $t_0 \geq i$ such that $\lim_s g(m_j, s, t) = 1$ for all $j < i$ and $\lim_s g(m_i, s, t) = 0$ whenever $t \geq t_0$.

Then for every leader $u \geq t_0$, $\mathscr{R}_i$ will claim u cofinitely many times, and therefore u will be in U if i is even and in $\overline{U}$ if i is odd. As every element follows the least leader below itself, every v above the least leader greater than t_0 will be in U if i is even and in $\overline{U}$ if i is odd. So if Z_e is infinite, there will be such a $v \in Z_e$ satisfying $\mathscr{R}_i$. Contradiction. □

□

5. Degrees bounding the Erdös Moser theorem

Another approach to the strength analysis of Ramsey's theorem for pairs consists in seeing a coloring $f : [\omega]^2 \to 2$ as an infinite tournament T such that $T(x, y)$ holds for $x < y$ if and only if $f(x, y) = 1$. The Erdős Moser theorem states the existence of an infinite transitive subtournament, that is, an infinite subset on which the tournament behaves like a linear order. Therefore the Erdős Moser theorem can be seen as a principle reducing instances of RT^2_2 into instances of ADS.

Definition 5.1 (Erdős Moser theorem) A tournament T on a domain $D \subseteq \mathbb{N}$ is an irreflexive binary relation on D such that for all $x, y \in D$ with $x \neq y$, exactly one of $T(x, y)$ or $T(y, x)$ holds. A tournament T is *transitive* if the corresponding relation T is transitive in the usual sense. A tournament T is *stable* if $(\forall x \in D)[(\forall^\infty s)T(x, s) \vee (\forall^\infty s)T(s, x)]$. EM is the statement "Every infinite tournament T has an infinite transitive subtournament." SEM is the restriction of EM to stable tournaments.

Bovykin and Weiermann proved in [2] that $\mathsf{EM} + \mathsf{ADS}$ is equivalent to RT^2_2 over RCA_0, equivalence still holding between their stable versions. Lerman et al. [18] proved over $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2$ that EM implies OPT and constructed an ω-model of EM that is not a model of SRT^2_2. Kreuzer proved in [17] that SEM implies $\mathsf{B}\Sigma^0_2$ over RCA_0. Bienvenu et al. proved in [1] that $\mathsf{RCA}_0 \vdash \mathsf{SEM} \to \mathsf{RWKL}$, hence there exists an ω-model of RRT^2_2 that is not a model of SEM. Wang constructed in [27] an ω-model of $\mathsf{EM} + \mathsf{COH}$ that is not a model of $\mathsf{STS}(2)$. Finally, the author proved in [23] that $\mathsf{RCA}_0 \vdash \mathsf{EM} \to [\mathsf{STS}(2) \vee \mathsf{COH}]$.

The following notion of *minimal interval* plays a fundamental role in the analysis of EM. See [18] for a background analysis of EM.

Definition 5.2 (Minimal interval) Let T be an infinite tournament and $a, b \in T$ be such that $T(a, b)$ holds. The *interval* (a, b) is the set of all $x \in T$ such that $T(a, x)$ and $T(x, b)$ hold. Let $F \subseteq T$ be a finite transitive subtournament of T. For $a, b \in F$ such that $T(a, b)$ holds, we say that (a, b) is a *minimal interval of* F if there is no $c \in F \cap (a, b)$, i.e., no $c \in F$ such that $T(a, c)$ and $T(c, b)$ both hold.

We provide in the next subsections two different proofs of the existence of a low_2 degree bounding EM. More precisely, we construct a low_2 set G which is, up to finite changes, transitive for every infinite computable tournament.

The author proved in [23] that $[\mathsf{STS}(2) \vee \mathsf{COH}] \leq_c \mathsf{EM}$. Therefore every low_2 degree bounding EM bounds also COH. The proof does not seem adaptable to prove that COH is a consequence of EM even in ω-models. However we can prove a weaker statement:

Lemma 5.3 For every set X, there exists an infinite X-computable tournament T such that for every infinite T-transitive subtournament U, $U \subseteq^* X$ or $U \subseteq^* \overline{X}$.

Proof. Fix a set X. We define a tournament T as follows: For each $a < b$, set $T(a, b)$ to hold iff $a \in X$ and $b \in X$ or $a \notin X$ and $b \notin X$. Suppose for the sake of absurd that U is an infinite transitive subtournament of T which intersects infinitely often X and $\overline{X}$. Take any $a, c \in U \cap X$ and $b, d \in U \cap \overline{X}$ such that $a < b < c < d$. Then $T(a, c)$, $T(c, b)$, $T(b, d)$ and $T(d, a)$ hold contradicting transitivity of U. $\square$

Using the previous lemma, the constructed set G must be cohesive and therefore provides another proof of the existence of a low_2 cohesive set. Finally, we can deduce a statement slightly weaker than Theorem 4.10 simply by the existence of a low_2 degree bounding EM.

Lemma 5.4 There exists a set C such that there is no low_2 over C degree $\mathbf{d} \gg_{\mathsf{SADS}} C$.

Proof. Fix a low_2 set $C \gg_{\mathsf{EM}} \emptyset$ and a set X low_2 over C. By low_2-ness of C, X is low_2. Consider the stable coloring $f : [\omega]^2 \to 2$ constructed by Mileti in [20, Corollary 5.4.5], such that X computes no infinite f-homogeneous set. We can see f as a stable tournament T such that for each $x < y$, $T(x, y)$ holds iff $f(x, y) = 1$. As $C \gg_{\mathsf{EM}} \emptyset$, there exists an infinite C-computable transitive subtournament U of T. U is a stable linear order such that every infinite ascending or descending sequence is f-homogeneous. Therefore X computes no infinite ascending or descending sequence in U. $\square$

The following question remains open:

Question 5.5 Does EM admit a universal instance?

5.1. **A low_2 degree bounding EM using first jump control.** The following theorem uses the proof techniques introduced in [4] for producing low_2 sets by controlling the first jump. It is done in the same spirit as Theorem 3.6 in [4].

Theorem 5.6 For every set $P \gg \emptyset'$, there exists a set $G \gg_{\mathsf{EM}} \emptyset$ such that $G' \leq_T P$.

Before proving Theorem 5.6, we introduce the notion of *Erdős Moser condition*.

Definition 5.7 An *Erdős Moser condition* (EM condition) for an infinite tournament T is a Mathias condition (F, X) where

(a) $F \cup \{x\}$ is T-transitive for each $x \in X$
(b) X is included in a minimal T-interval of F.

Extension is usual Mathias extension. EM conditions have good properties for tournaments as stated by the following lemmas. Given a tournament T and two sets E and F, we denote by $E \to_T F$ the formula $(\forall x \in E)(\forall y \in F)T(x,y)$ holds.

Lemma 5.8 Fix an EM condition (F, X) for a tournament T. For every $x \in F$, $\{x\} \to_T X$ or $X \to_T \{x\}$.

Proof. Fix an $x \in F$. Let (u, v) be the minimal T-interval containing X, where u, v may be respectively $-\infty$ and $+\infty$. By definition of interval, $\{u\} \to_T X \to_T \{v\}$. By definition of minimal interval, $T(x,u)$ or $T(v,x)$ holds. Suppose the former holds. By transitivity of $F \cup \{y\}$ for every $y \in X$, $T(x,y)$ holds, therefore $\{x\} \to_T X$. In the latter case, by symmetry, $X \to_T \{x\}$. □

Lemma 5.9 Fix an EM condition $c = (F, X)$ for a tournament T, an infinite subset $Y \subseteq X$ and a finite T-transitive set $F_1 \subset X$ such that $F_1 < Y$ and $[F_1 \to_T Y \vee Y \to_T F_1]$. Then $d = (F \cup F_1, Y)$ is a valid extension of c.

Proof. Properties of a Mathias condition for d are immediate. We prove property (a). Fix an $x \in Y$. To prove that $F \cup F_1 \cup \{x\}$ is T-transitive, it suffices to check that there exists no 3-cycle in $F \cup F_1 \cup \{x\}$. Fix three elements $u < v < w \in F \cup F_1 \cup \{x\}$.

- Case 1: $\{u, v, w\} \cap F \neq \emptyset$. Then $u \in F$ as $F < F_1 < \{x\}$ and $u < v < w$. If $v \in F$ then using the fact that $F_1 \cup \{x\} \subset X$ and property (a) of condition c, $\{u, v, w\}$ is T-transitive. If $v \notin F$, then by Lemma 5.8, $\{u\} \to_T X (\supseteq F \cup \{x\})$ or $X \to_T \{u\}$ hence $\{u\} \to_T \{v, w\}$ or $\{v, w\} \to_T \{u\}$ so $\{u, v, w\}$ is T-transitive.
- Case 2: $\{u, v, w\} \cap F = \emptyset$. Then at least $u, v \in F_1$ because $F_1 < \{x\}$. If $w \in F_1$, then $\{u, v, w\}$ is T-transitive by T-transitivity of F_1. Otherwise, as $F_1 \to_T Y$ or $Y \to_T F_1$, $\{u, v\} \to_T \{w\}$ or $\{w\} \to_T \{u, v\}$ and $\{u, v, w\}$ is T-transitive.

We now prove property (b). Let (u, v) be the minimal T-interval of F in which X (hence Y) is included by property (b) of condition c. u and v may be respectively $-\infty$ and $+\infty$. By assumption, either $F_1 \to_T Y$ or $Y \to_T F_1$. As F_1 is a finite T-transitive set, it has a minimal and a maximal element, say x and y. If $F_1 \to_T Y$ then Y is included in the T-interval (y, v). Symmetrically, if $Y \to_T F_1$ then Y is included in the T-interval (u, x). To prove minimality for the first case, assume that some w is in the interval (y, v). Then $w \notin F$ by minimality of the interval (u, v) w.r.t. F, and $w \notin F_1$ by maximality of y. Minimality for the second case holds by symmetry. □

Proof of Theorem 5.6. Let C be a low set such that there exists a uniformly C-computable enumeration $\vec{T}$ of infinite tournaments containing every computable tournament. Note that $P \gg C'$. Our forcing conditions are tuples (σ, F, X) where $\sigma \in \omega^{<\omega}$ and the following holds:

(a) (F,X) forms a Mathias condition and X is a set low over C.
(b) $(F \setminus [0,\sigma(\nu)],X)$ is an EM condition for T_ν for each $\nu < |\sigma|$.

A condition $(\tilde{\sigma}, \tilde{F}, \tilde{X})$ *extends* a condition (σ, F, X) if $\sigma \preceq \tilde{\sigma}$ and $(\tilde{F}, \tilde{X})$ Mathias extends (F,X). A set G *satisfies* the condition (σ, F, X) if $G \setminus [0,\sigma(\nu)]$ is T_ν-transitive for each $\nu < |\sigma|$ and G satisfies the Mathias condition (F,X). An *index* of a condition (σ, F, X) is a code of the tuple $\langle \sigma, F, e\rangle$ where e is a lowness index of X.

The first lemma simply states that we can ensure that G will be infinite and eventually transitive for each tournament in $\vec{T}$.

Lemma 5.10 For every condition $c = (\sigma, F, X)$ and every $i, j \in \omega$, one can P-compute an extension $(\tilde{\sigma}, \tilde{F}, \tilde{X})$ such that $|\tilde{\sigma}| \geq i$ and $|\tilde{F}| \geq j$ uniformly from i, j and an index of c.

Proof. Let x be the first element of X. As X is low over C, x can be found C'-computably from a lowness index of X. The condition $(\tilde{\sigma}, F, X)$ is a valid extension of c where $\tilde{\sigma} = \sigma^\frown x \dots x$ so that $|\tilde{\sigma}| \geq i$. It suffices to prove that we can C'-compute an extension $(\tilde{\sigma}, \tilde{F}, \tilde{X})$ with $|\tilde{F}| > |F|$ and iterate the process. Define the computable coloring $g : X \to 2^{|\tilde{\sigma}|}$ by $g(s) = \rho$ where $\rho \in 2^{|\tilde{\sigma}|}$ such that $\rho(\nu) = 1$ iff $T_\nu(x,s)$ holds. One can find uniformly in P a $\rho \in 2^{|\tilde{\sigma}|}$ such that the following C-computable set is infinite:

$$Y = \{s \in X \setminus \{x\} : g(s) = \rho\}$$

By Lemma 5.9, $((F \cup \{x\}) \setminus [0, \tilde{\sigma}(\nu)], Y)$ is a valid EM extension for T_ν. As Y is low over C, $(\tilde{\sigma}, F \cup \{x\}, Y)$ is a valid extension for c. $\square$

It remains to be able to decide $e \in (G \oplus C)'$ uniformly in e. We first need to define a forcing relation.

Definition 5.11 Fix a condition $c = (\sigma, F, X)$ and two integers e and x.

1. $c \Vdash \Phi_e^{G \oplus C}(x) \uparrow$ if $\Phi_e^{(F \cup F_1) \oplus C}(x) \uparrow$ for all finite subsets $F_1 \subseteq X$ such that F_1 is T_ν-transitive simultaneously for each $\nu < |\sigma|$.
2. $c \Vdash \Phi_e^{G \oplus C}(x) \downarrow$ if $\Phi_e^{F \oplus C}(x) \downarrow$.

Note that the way we defined our forcing relation $c \Vdash \Psi_e^{G \oplus C}(x) \uparrow$ differs slightly from the "true" forcing notion $\Vdash^*$ inherited by the notion of satisfaction of G. The true forcing definition of this statement is the following:

$c \Vdash^* \Phi_e^{G \oplus C}(x) \uparrow$ if $\Phi_e^{(F \cup F_1) \oplus C}(x) \uparrow$ for all finite *extensible* subsets $F_1 \subseteq X$ such that F_1 is T_ν-transitive simultaneously for each $\nu < |\sigma|$, i.e., for all finite subsets $F_1 \subseteq X$ such that there exists an extension $d = (\tilde{\sigma}, F \cup F_1, \tilde{X})$.

However $c \Vdash^* \Phi_e^{G \oplus C}(x) \uparrow$ is not a Π^0_1 statement whereas $c \Vdash \Phi_e^{G \oplus C}(x) \uparrow$ is. In particular the fact that $c \nVdash \Phi_e^{G \oplus C}(x) \uparrow$ does not mean that c has an extension forcing its negation. This subtlety is particularly important in Lemma 5.13. The following lemma gives a sufficient constraint, namely being included in a part of a particular partition, on finite transitive sets to ensure that they are *extensible*.

Lemma 5.12 Let $c = (\sigma, F, X)$ be a condition and $E \subseteq X$ be a finite set. There exists a $2^{|\sigma|}$ partition $(E_\rho : \rho \in 2^{|\sigma|})$ of E and an infinite set $Y \subseteq X$ low over C such that $E < Y$ and for all $\rho \in 2^{|\sigma|}$ and $\nu < |\sigma|$, if $\rho(\nu) = 0$ then $E_\rho \to_{T_\nu} Y$ and if $\rho(\nu) = 1$ then $Y \to_{T_\nu} E_\rho$.

Moreover this partition and a lowness index of Y can be uniformly P-computed from an index of c and the set E.

Proof. Given a set E, define P_E to be the finite set of ordered $2^{|\sigma|}$-partitions of E, that is,

$$P_E = \{(E_\rho : \rho \in 2^{|\sigma|}) : \bigcup_{\rho \in 2^{|\sigma|}} E_\rho = E \text{ and } \rho \neq \xi \rightarrow E_\rho \cap E_\xi = \emptyset\}$$

Define the C-computable coloring $g : X \to P_E$ by $g(x) = (E^x_\rho : \rho \in 2^{|\sigma|})$ where $E^x_\rho = \{a \in E : (\forall \nu < |\sigma|)[T_\nu(a,x) \text{ holds iff } \rho(\nu) = 0]\}$. On can find uniformly in P a partition $(E_\rho : \rho \in 2^{|\sigma|})$ such that the following C-computable set is infinite:

$$Y = \{x \in X \setminus E : g(x) = (E_\rho : \rho \in 2^{|\sigma|})\}$$

By definition of g, for all $\rho \in 2^{|\sigma|}$ and $\nu < |\sigma|$, if $\rho(\nu) = 0$ then $E_\rho \rightarrow_{T_\nu} Y$ and if $\rho(\nu) = 1$ then $Y \rightarrow_{T_\nu} E_\rho$. □

We are now ready to prove the key lemma of this forcing, stating that we can P-decide whether or not $e \in G'$ for any $e \in \omega$.

Lemma 5.13 For every condition (σ, F, X) and every $e \in \omega$, there exists an extension $d = (\tilde{\sigma}, \tilde{F}, \tilde{X})$ such that one of the following holds:

1. $d \Vdash \Phi_e^{G \oplus C}(e) \downarrow$
2. $d \Vdash \Phi_e^{G \oplus C}(e) \uparrow$

This extension can be P-computed uniformly from an index of c and e. Moreover there is a C'-computable procedure to decide which case holds from an index of d.

Proof. Let $k = |\sigma|$. Using a C'-computable procedure, we can decide from an index of c and e whether there exists a finite set $E \subset X$ such that for every 2^k-partition $(E_i : i < 2^k)$ of E, there exists an $i < 2^k$ and a subset $F_1 \subseteq E_i$ T_ν-transitive simultaneously for each $\nu < k$ and satisfying $\Phi_e^{(F \cup F_1) \oplus C}(e) \downarrow$.

1. If such a set E exists, it can be C'-computably found. By Lemma 5.12, one can P-computably find a 2^k-partition $(E_\rho : \rho \in 2^k)$ of E and a set $Y \subseteq X$ low over C such that for all $\rho \in 2^k$ and $\nu < k$, if $\rho(\nu) = 0$ then $E_\rho \rightarrow_{T_\nu} Y$ and if $\rho(\nu) = 1$ then $Y \rightarrow_{T_\nu} E_\rho$. We can C'-computably find a $\rho \in 2^k$ and a set $F_1 \subseteq E_\rho$ which is T_ν-transitive simultaneously for each $\nu < k$ and satisfying $\Phi_e^{(F \cup F_1) \oplus C}(e) \downarrow$. By Lemma 5.9, $(F \setminus [0, \sigma(\nu)]) \cup F_1, Y)$ is a valid EM extension of $(F \setminus [0, \sigma(\nu)], X)$ for T_ν for each $\nu < k$. As Y is low over C, $(\sigma, F \cup F_1, Y)$ is a valid extension of c forcing $\Phi_e^{G \oplus C}(e) \downarrow$.
2. If no such set exists, then by compactness, the $\Pi_1^{0,C}$ class of all 2^k-partitions $(X_i : i < 2^k)$ of X such that for every $i < 2^k$ and every finite set $F_1 \subseteq X_i$ which is T_ν-transitive simultaneously for each $\nu < k$, $\Phi_e^{(F \cup F_1) \oplus C}(e) \uparrow$ is non-empty. In other words, the $\Pi_1^{0,C}$ class of all 2^k-partitions $(X_i : i < 2^k)$ of X such that for every $i < 2^k$, $(\sigma, F, X_i) \Vdash \Phi_e^{G \oplus C}(e) \uparrow$ is non-empty. By the relativized low basis theorem, there exists a 2^k-partition $(X_i : i < 2^k)$ of X low over C. Furthermore, a lowness index for this partition can be uniformly C'-computably found. Using P, one can find an $i < 2^k$ such that X_i is infinite. (σ, F, X_i) is a valid extension of c forcing $\Phi_e^{G \oplus C}(e) \uparrow$.

□

Using Lemma 5.10 and Lemma 5.13, one can P-compute an infinite decreasing sequence of conditions $c_0 = (\epsilon, \emptyset, \omega) \geq c_1 \geq \dots$ such that for each $s > 0$

1. $|\sigma_s| \geq s$, $|F_s| \geq s$
2. $c_s \Vdash \Phi_s^{G \oplus C}(s) \downarrow$ or $c_s \Vdash \Phi_s^{G \oplus C}(s) \uparrow$

where $c_s = (\sigma_s, F_s, X_s)$. The resulting set $G = \bigcup_s F_s$ is T_ν-transitive up to finite changes for each $\nu \in \omega$ and $G' \leq_T P$. □

5.2. **A low$_2$ degree bounding EM using second jump control.** We now use the second proof technique used in [4] for producing a low$_2$ set. It consists of directly controlling the second jump of the produced set.

Theorem 5.14 There exists a low$_2$ degree bounding EM.

Proof. Similar to Theorem 5.6, we fix a low set C such that there exists a uniformly C-computable enumeration $\vec{T}$ of infinite tournaments containing every computable tournament. In particular $P \gg C'$.

Our forcing conditions are the same as in Theorem 5.6. We can release the constraints of infinity and lowness over C for X in a condition (σ, F, X). This gives the notion of *precondition*. The forcing relations extend naturally to preconditions.

Definition 5.15 Fix a finite set of Turing indexes $\vec{e}$. A condition (σ, F, X) is *$\vec{e}$-small* if there exists a number x and a sequence $(\sigma_i, F_i, X_i : i < n)$ such that for each $i < n$

(i) (σ_i, F_i, X_i) is a precondition extending c
(ii) $(X_i : i < n)$ is a partition of $X \cap (x, +\infty)$
(iii) $max(X_i) < x$ or $(\sigma_i, F \cup F_i, X_i) \Vdash (\exists e \in \vec{e})(\exists y < x)\Phi_e^{G \oplus C}(y) \uparrow$

A condition is *$\vec{e}$-large* if it is not $\vec{e}$-small.

A condition $(\tilde{\sigma}, \tilde{F}, \tilde{X})$ is a *finite extension* of (σ, F, X) if $\tilde{X} =^* X$. Finite extensions do not play the same fundamental role as in the original forcing in [4] as adding elements to the set F may require to remove infinitely many elements of the promise set X to obtain a valid extension. We nevertheless prove the following traditional lemma.

Lemma 5.16 Fix an $\vec{e}$-large condition $c = (\sigma, F, X)$.

1. If $\vec{e'} \subseteq \vec{e}$ then c is $\vec{e'}$-large.
2. If d is a finite extension of c then d is $\vec{e}$-large.

Proof. Clause 1 is trivial as $\vec{e}$ appears only in a universal quantification in the definition of $\vec{e}$-largeness. We prove clause 2. Let $d = (\tilde{\sigma}, \tilde{F}, \tilde{X})$ be an $\vec{e}$-small finite extension of c. We will prove that c is $\vec{e}$-small. Let $x \in \omega$ and $(\sigma_i, F_i, X_i : i < n)$ witness $\vec{e}$-smallness of d. Let $y = max(x, X \setminus \tilde{X})$. For each $i < n$, set $\tilde{X}_i = X_i \cap (y, +\infty)$. Then y and $(\sigma_i, F_i, \tilde{X}_i : i < n)$ witness $\vec{e}$-smallness of c. □

Lemma 5.17 There exists a C''-effective procedure to decide, given an index of a condition c and a finite set of Turing indexes $\vec{e}$, whether c is $\vec{e}$-large. Furthermore, if c is $\vec{e}$-small, there exists sets $(X_i : i < n)$ low over C witnessing this, and one may C'-compute a value of n, x, lowness indexes for $(X_i : i < n)$ and the corresponding sequences $(\sigma_i, F_i, X_i : i < n)$ which witness that c is $\vec{e}$-small.

Proof. Fix a condition $c = (\sigma, F, X)$ The predicate "(σ, F, X) is $\vec{e}$-small" can be expressed as a Σ^0_2 statement

$$(\exists z)(\exists Z)P(z, Z, F, X, \vec{\nu}, \vec{e})$$

where P is a $\Pi^{0,C}_1$ predicate. Here z codes n and x, and Z codes $(X_i : i < n)$. The predicate $(\exists Z)P(z, Z, F, X, \sigma, \vec{e})$ is $\Pi^{0,C\oplus X}_1$ by compactness. As X is low over C and F and σ are finite, one can compute a $\Delta^{0,C}_2$ index for the same predicate P with parameter z, an index of c and $\vec{e}$, from a lowness index for X, F and σ. Therefore there exists a $\Sigma^{0,C}_2$ statement with parameters an index of c and $\vec{e}$ which holds iff c is $\vec{e}$-small.

If c is $\vec{e}$-small, there exists sets $(X_i : i < n)$ low over X (hence low over C) witnessing it by the low basis theorem relativized to C. By the uniformity of the proof of the low basis theorem, one can compute lowness indexes of $(X_i : i < n)$ uniformly from a lowness index of X. ☐

As the extension produced in Lemma 5.10 is not a finite extension, we need to refine it to ensure largeness preservation.

Lemma 5.18 For every $\vec{e}$-large condition $c = (\sigma, F, X)$ and every $i, j \in \omega$, one can P-compute an $\vec{e}$-large extension $(\tilde{\sigma}, \tilde{F}, \tilde{X})$ such that $\tilde{\sigma} \geq i$ and $|\tilde{F}| \geq j$ uniformly from an index of c, i, j and $\vec{e}$.

Proof. Let x be the first element of X. As X is low over C, x can be found C'-computably from a lowness index of X. The condition $d = (\tilde{\sigma}, F, X)$ is a valid extension of c where $\tilde{\sigma} = \sigma^\frown x \dots x$ so that $|\tilde{\sigma}| \geq i$. As d is a finite extension of c, it is $\vec{e}$-large by Lemma 5.16. It suffices to prove that we can C'-compute an $\vec{e}$-large extension $(\tilde{\sigma}, \tilde{F}, \tilde{X})$ with $|\tilde{F}| > |F|$ and iterate the process. Define the C-computable coloring $g : X \to 2^{|\tilde{\sigma}|}$ as in Lemma 5.10. For each $\rho \in 2^{|\tilde{\sigma}|}$, define the following set:

$$Y_\rho = \{s \in X \setminus \{x\} : g(s) = \rho\}$$

There must be a $\rho \in 2^{|\tilde{\sigma}|}$ such that Y_ρ is infinite and $(\tilde{\sigma}, F \cup \{x\}, Y_\rho)$ is $\vec{e}$-large, otherwise the witnesses of $\vec{e}$-smallness for each $\rho \in 2^{|\tilde{\sigma}|}$ would witness $\vec{e}$-smallness of c. By Lemma 5.17, one can C''-find a $\rho \in 2^{|\tilde{\sigma}|}$ such that $(\tilde{\sigma}, F \cup \{x\}, Y_\rho)$ is $\vec{e}$-large. As seen in Lemma 5.10, $(\tilde{\sigma}, F, \{x\}, Y_\rho)$ is a valid extension. ☐

The following lemma is a refinement of Lemma 5.12 controlling largeness preservation.

Lemma 5.19 Let $c = (\sigma, F, X)$ be an $\vec{e}$-large condition and $E \subseteq X$ be a finite set. There is a $2^{|\sigma|}$ partition $(E_\rho : \rho \in 2^{|\sigma|})$ of E and an infinite set $Y \subseteq X$ low over C such that $E < Y$ and

1. for all $\rho \in 2^{|\sigma|}$ and $\nu < |\sigma|$, if $\rho(\nu) = 0$ then $E_\rho \to_{T_\nu} Y$ and if $\rho(\nu) = 1$ then $Y \to_{T_\nu} E_\rho$.
2. $(\sigma, F \cup F_1, Y)$ is an $\vec{e}$-large condition extending c for every $\rho \in 2^{|\sigma|}$ and every finite set $F_1 \subseteq E_\rho$ which is T_ν-transitive for each $\nu < |\sigma|$

Moreover this partition and a lowness index of Y can be uniformly C''-computed from an index of c and the set E.

Proof. Given a set E, recall from Lemma 5.12 that P_E is the finite set of ordered 2^k-partitions of E. Define again the computable coloring $g : X \to P_E$ by $g(x) = (E^x_\rho : \rho \in 2^{|\sigma|})$ where $E^x_\rho = \{a \in E : (\forall \nu < |\sigma|)[T_\nu(a, x) \text{ holds iff } \rho(\nu) = 0]\}$. If for each partition $(E_\rho : \rho \in 2^{|\sigma|})$, there exists a $\rho \in 2^{|\sigma|}$ and a $F_1 \subseteq E_\rho$ which is T_ν-transitive simultaneously for each $\nu < |\sigma|$ and such that $(\sigma, F \cup F_1, Y)$ is $\vec{e}$-small where

$$Y = \{x \in X \setminus E : g(x) = (E_\rho : \rho \in 2^{|\sigma|})\}$$

Then we could construct a witness of $\vec{e}$-smallness of c using smallness witnesses of $(\sigma, F \cup F_1, Y)$ for each partition $(E_\rho : \rho \in 2^{|\sigma|})$. Therefore there must exist a partition $(E_\rho : \rho \in 2^{|\sigma|})$ such that Y is infinite and $d = (\sigma, F \cup F_1, Y)$ is $\vec{e}$-large for every $\rho \in 2^{|\sigma|}$ and every $F_1 \subseteq E_\rho$ which is T_ν-transitive for each $\nu < |\sigma|$.

By Lemma 5.17, such a partition can be found C''-computably. By definition of g, for all $\rho \in 2^{|\sigma|}$ and $\nu < k$, if $\rho(\nu) = 0$ then $E_\rho \to_{T_\nu} Y$ and if $\rho(\nu) = 1$ then $Y \to_{T_\nu} E_\rho$. Therefore, by Lemma 5.9, $((F \setminus [0, \sigma(\nu)]) \cup F_1, Y)$ is a valid EM extension of $(F \setminus [0, \sigma(\nu)], X)$ for T_ν for each $\nu < |\sigma|$, so d is a valid condition. ☐

Lemma 5.20 Suppose that $c = (\sigma, F, X)$ is $\vec{e}$-large. For every $y \in \omega$ and $e \in \vec{e}$, there exists an $\vec{e}$-large extension d such that $d \Vdash \Phi_e^{G \oplus C}(y)\downarrow$. Furthermore, an index for d can be computed from an oracle for C' from an index of c, e and y.

Proof. Let $k = |\sigma|$. As c is $\vec{e}$-large, then by a compactness argument, there exists a finite set $E \subset X$ such that for every 2^k-partition $(E_i : i < 2^k)$ of E, there exists an $i < k$ and a finite subset $F_1 \subseteq E_i$ which is T_ν-transitive simultaneously for each $\nu < k$, and $\Phi_e^{(F \cup F_1) \oplus C}(y)\downarrow$. Moreover this set E can be C'-computably found. By Lemma 5.19, on can uniformly C''-find a partition $(E_\rho : \rho \in 2^k)$ of E and a lowness index for an infinite set $Y \subseteq X$ low over C such that

1. for all $\rho \in 2^k$ and $\nu < k$, if $\rho(\nu) = 0$ then $E_\rho \to_{T_\nu} Y$ and if $\rho(\nu) = 1$ then $Y \to_{T_\nu} E_\rho$.
2. $(\sigma, F \cup F_1, Y)$ is an $\vec{e}$-large condition extending c for every $\rho \in 2^k$ and every finite set finite set $F_1 \subseteq E_\rho$ which is T_ν-transitive for each $\nu < k$

We can then produce by a C'-computable search a $\rho \in 2^k$ and a finite set $F_1 \subseteq E_\rho$ which is T_ν-transitive for each $\nu < k$ and such that $\Phi_e^{(F \cup F_1) \oplus C}(y)\downarrow$. By Lemma 5.9, $((F \setminus [0, \sigma(\nu)]) \cup F_1, Y)$ is a valid EM extension of $(F \setminus [0, \sigma(\nu)], X)$ for T_ν for each $\nu < k$. As Y is low over C, $(\sigma, F \cup F_1, Y)$ is a valid $\vec{e}$-large extension. ☐

Lemma 5.21 Suppose that $c = (\sigma, F, X)$ is $\vec{e}$-large and $(\vec{e} \cup \{u\})$-small. There exists a $\vec{e}$-large extension d such that $d \Vdash \Phi_u^{G \oplus C}(y)\uparrow$ for some $y \in \omega$. Furthermore one can find an index for d by applying a C''-computable function to an index of c, $\vec{e}$ and u.

Proof. By Lemma 5.17, we may choose the sets $(X_i : i < n)$ witnessing that c is $(\vec{e} \cup \{u\})$-small to be low over C. Fix the corresponding x and $(\sigma_i, F_i : i < n)$. Consider the i's such that $(\sigma_i, F_i, X_i) \Vdash \Phi_u^{G \oplus C}(y)\uparrow$ for some $y < x$. As c is $\vec{e}$-large, there must be such an $i < n$ such that (σ_i, F_i, X_i) is an $\vec{e}$-large condition. By Lemma 5.17 we can find C''-computably such an $i < n$. (σ_i, F_i, X_i) is the desired extension. ☐

Using the previous lemmas, we can C''-compute an infinite descending sequence of conditions $c_0 = (\epsilon, \emptyset, \omega) \geq c_1 \geq \dots$ together with an infinite increasing sequence of Turing indexes $\vec{e}_0 = \emptyset \subseteq \vec{e}_1 \subseteq \dots$ such that for each $s > 0$

1. $|\sigma_s| \geq s$, $|F_s| \geq s$, c_s is $\vec{e}_s$-large
2. Either $s \in \vec{e}_s$ or $c_s \Vdash \Phi_s^{G \oplus C}(y)\uparrow$ for some $y \in \omega$
3. $c_s \Vdash \Phi_e^{G \oplus C}(x)\downarrow$ if $s = \langle e, x \rangle$ and $e \in \vec{e}_s$

where $c_s = (\sigma_s, F_s, X_s)$. The resulting set $G = \bigcup_s F_s$ is T_ν-transitive up to finite changes simultaneously for each $\nu \in \omega$ and $G'' \leq_T C'' \leq_T \emptyset''$. $\square$

6. Degree bounding the rainbow Ramsey theorem

The rainbow Ramsey theorem intuitively states that when a coloring over tuples uses each color a bounded number of times then it has an infinite subset on which each color is used at most once. This statement has been extensively studied over the past few years [8, 7, 26, 23]. Remarkably, the restriction of the rainbow Ramsey theorem to coloring over pairs of integers coincides with a well-known notion of algorithmic randomness.

Definition 6.1 (Rainbow Ramsey theorem) Let $n, k \in \omega$. A coloring function $f : [\omega]^n \to \omega$ is *k-bounded* if for every $y \in \omega$, $\left|f^{-1}(y)\right| \leq k$. A set R is a *rainbow* for f if $f \restriction [R]^n$ is injective. RRT^n_k is the statement "Every k-bounded function $f : [\omega]^n \to \omega$ has an infinite rainbow".

A proof of the rainbow Ramsey theorem is due to Galvin who noticed that it follows easily from Ramsey's theorem. Hence every computable 2-bounded coloring function f over n-tuples has an infinite Π^0_n rainbow. Csima and Mileti proved in [8] that every 2-random is RRT^2_2-bounding and deduced that RRT^2_2 implies neither SADS nor WKL_0 over ω-models. Conidis & Slaman adapted in [7] the argument from Cisma and Mileti to obtain $\mathsf{RCA}_0 \vdash \text{2-}\mathsf{RAN} \to \mathsf{RRT}^2_2$.

Definition 6.2 A function $f : \omega \to \omega$ is *diagonally non-computable (DNC) relative to X* if $f(e) \neq \Phi^X_e(e)$ for each $e \in \omega$. $\mathsf{DNR}[\emptyset']$ is the statement "For every set X, there exists a function DNC relative to the jump of X".

Theorem 6.3 (J.S. Miller [21]) RRT^2_2 and $\mathsf{DNR}[\emptyset']$ are computably equivalent.

Corollary 6.4 RRT^2_2 admits a universal instance.

Proof. If P and Q are two principles computably equivalent and Q admits a universal instance, then so does P. As $\mathsf{DNR}[\emptyset']$ admits a universal instance (any function DNC relative to $\emptyset'$), so does RRT^2_2. $\square$

Corollary 6.5 For every $X \gg \emptyset'$, there exists a $Y \gg_{\mathsf{RRT}^2_2} \emptyset$ such that $Y' \leq_T X$.

Proof. Let $f : [\omega]^2 \to \omega$ be a universal instance of RRT^2_2. By Csima & Mileti [8], $\mathsf{RRT}^2_2 \leq_c \mathsf{RT}^2_2$, so there exists a computable coloring $g : [\omega]^2 \to 2$ such that every infinite g-homogeneous set computes an infinite f-rainbow, hence bounds RRT^2_2. By Cholak et al. [4], for every $X \gg \emptyset'$ there exists an infinite g-homogeneous set H such that $H' \leq_T X$. In particular $H \gg_{\mathsf{RRT}^2_2} \emptyset$. $\square$

Corollary 6.6 There exists a low$_2$ degree bounding RRT^2_2.

Proof. By the relativized low basis theorem, there exists a set $X \gg \emptyset'$ low over $\emptyset'$. By Corollary 6.5, there exists a set $Y \gg_{\mathsf{RRT}^2_2} \emptyset$ such that $Y' \leq_T X$, hence $Y'' \leq_T X' \leq_T \emptyset''$. So Y is low$_2$. $\square$

We can generalize Corollary 6.6 to colorings over arbitrary tuples. For this, we need to restrict ourselves to the study of a particular class of colorings.

Definition 6.7 A coloring $f : [\omega]^{n+1} \to \omega$ is *normal* if $f(\sigma, a) \neq f(\tau, b)$ for each $\sigma, \tau \in [\omega]^n$, whenever $a \neq b$.

Wang proved in [26] that for every 2-bounded coloring $f : [\omega]^n \to \omega$, every f-random computes an infinite set X on which f is normal. The author refined in [23] this result by proving that every function d.n.c. relative to f computes such a set.

Theorem 6.8 For each $n \geq 0$, there exists a set $X \gg_{\mathsf{RRT}^{n+2}_2} \emptyset$ low$_2$ over $\emptyset^{(n)}$.

Proof. We prove by induction over n that for every set A there exists a set X low$_2$ over $A^{(n)}$ such that $X \gg_{\mathsf{RRT}^{n+2}_2} A$. Case $n = 0$ is a relativization of Corollary 6.6. Suppose for each set A, there exists a set X low$_2$ over $A^{(n)}$ such that $X \gg_{\mathsf{RRT}^{n+2}_2} A$. Fix a set A, an A-random set R low over A and a set C low$_2$ over $A \oplus R$ such that $C' \gg (A \oplus R)'$. In particular $R \oplus C$ is low$_2$ over A. By induction hypothesis, there exists a set X low$_2$ over $(A \oplus R \oplus C)^{(n+1)}$ such that $X \gg_{\mathsf{RRT}^{n+2}_2} (A \oplus R \oplus C)'$. In particular X is low$_2$ over $A^{(n+1)}$. We can assume without loss of generality that X computes A, since $X \oplus A$ is low$_2$ over $A^{(n+1)}$ and $X \oplus A \gg_{\mathsf{RRT}^{n+2}_2} (A \oplus R \oplus C)'$. We claim that $X \gg_{\mathsf{RRT}^{n+3}_2} A$.

Fix an A-computable 2-bounded coloring $f : [\omega]^{n+3} \to \omega$. By relativizing Lemma 4.3 in [26], every A-random computes an infinite set Y such that f restricted to Y is normal. So $A \oplus R$ computes such a set Y. For each $\sigma, \tau \in [Y]^{n+2}$, let

$$U_{\sigma,\tau} = \{s \in Y : f(\sigma, s) = f(\tau, s)\}$$

By Jockusch & Frank [16], as $C' \gg (A \oplus R)'$, $A \oplus R \oplus C$ computes an infinite $\vec{U}$-cohesive set $Z \subseteq Y$. In particular the following limit exists

$$\tilde{f}(\sigma) = \lim_{s \in Z} min\{\tau \leq_{lex} \sigma : f(\sigma, s) = f(\tau, s)\}$$

$\tilde{f}$ is a 2-bounded $(A \oplus R \oplus C)'$-computable coloring of $(n+2)$-tuples, so X bounds an infinite $\tilde{f}$-rainbow H. $A \oplus H$ computes an infinite f-rainbow, so X bounds an infinite f-rainbow. □

6.1. **A stable rainbow Ramsey theorem.** A common process in the strength analysis of a principle consists of splitting the statement into a stable and a cohesive version. The standard notion of stability does not apply for the rainbow Ramsey theorem as no stable coloring is k-bounded for some $k \in \omega$. Nevertheless one can define certain notions of stability for the rainbow Ramsey theorem [23]. Mileti proved in [20] that the only Δ^0_2 degree bounding SRT^2_2 is $\mathbf{0}'$. In fact, his priority argument can be adapted to prove the same result on a much weaker principle coinciding with a stable version of the rainbow Ramsey theorem for pairs.

Definition 6.9 A coloring $f : [\omega]^2 \to \omega$ is *rainbow-stable* if for every $x \in \omega$, one of the following holds:

(a) There exists a $y \neq x$ such that $(\forall^\infty s) f(x,s) = f(y,s)$
(b) $(\forall^\infty s)\, |\{y \neq x : f(x,s) = f(y,s)\}| = 0$

SRRT^2_2 is the statement “every rainbow-stable 2-bounded coloring $f : [\omega]^2 \to \omega$ has a rainbow.”

Introduced by the author in [23], he proved that SRRT^2_2 is computably reducible to SEM and $\mathsf{STS}(2)$. This principle admits many computably equivalent formulations. We are particularly interested in a characterization which can be seen as a stable notion of $\mathsf{DNR}[\emptyset']$.

Definition 6.10 Given a function $f : \omega \to \omega$, a function g is *f-diagonalizing* if $(\forall x)[f(x) \neq g(x)]$. $\mathsf{SDNR}[\emptyset']$ is the statement "Every Δ^0_2 function $f : \omega \to \omega$ has an f-diagonalizing function".

Theorem 6.11 (Patey [23]) SRRT^2_2 and $\mathsf{SDNR}[\emptyset']$ are computably equivalent.

The following theorem extends Mileti's result to $\mathsf{SDNR}[\emptyset']$. As $\mathsf{SDNR}[\emptyset']$ is computably below many stable principles, we shall deduce a few more non-universality results.

Theorem 6.12 For every Δ^0_2 incomplete set X, there exists a Δ^0_2 function $f : \omega \to \omega$ such that X computes no f-diagonalizing function.

Corollary 6.13 A Δ^0_2 degree $\mathbf{d}$ bounds SRRT^2_2 iff $\mathbf{d} = \mathbf{0}'$.

Proof. As $\mathsf{SRRT}^2_2 \leq_c \mathsf{SRT}^2_2$, any computable instance of SRRT^2_2 has a Δ^0_2 solution. So $\mathbf{0}'$ bounds SRRT^2_2. If $\mathbf{d}$ is incomplete, then by Theorem 6.12 and by $\mathsf{SRRT}^2_2 =_c \mathsf{SDNR}[\emptyset']$, there is a computable instance of SRRT^2_2 such that $\mathbf{d}$ bounds no solution. □

Corollary 6.14 No statement P such that $\mathsf{SRRT}^2_2 \leq_c \mathsf{P} \leq_c \mathsf{SRT}^2_2$ admits a universal instance.

Proof. By [13, Corollary 4.6] every Δ^0_2 set or its complement has an incomplete Δ^0_2 infinite subset. As $\mathsf{P} \leq_c \mathsf{SRT}^2_2 \leq_c \mathsf{D}^2_2$, every computable instance U of $\mathscr{P}$ has a Δ^0_2 incomplete solution X. By Theorem 6.12, there exists a computable coloring $f : [\omega]^2 \to \omega$ such that X computes no infinite f-rainbow. As $\mathsf{SRRT}^2_2 \leq_c \mathsf{P}$, there exists a computable instance of P such that X does not compute a solution to it. Hence U is not a universal instance of P. □

Corollary 6.15 None of SRRT^2_2, SEM, $\mathsf{STS}(2)$ and $\mathsf{SFS}(2)$ admits a universal instance.

Proof of Theorem 6.12. The proof is an adaptation of [20, Theorem 5.3.7]. Suppose that D is a Δ^0_2 incomplete set. We will construct a Δ^0_2 function $f : \omega \to \omega$ such that D does not compute any f-diagonalizing function. We want to satisfy the following requirements for each $e \in \omega$:

$$\mathscr{R}_e : \text{If } \Phi^D_e \text{ is total, then there is an } a \text{ such that } \Phi^D_e(a) = f(a).$$

For each $e \in \omega$, define the partial function u_e by letting $u_e(a)$ be the use of Φ^D_e on input a if $\Phi^D_e(a)\downarrow$ and letting $u_e(a)\uparrow$ otherwise. We can assume w.l.o.g. that whenever $u_e(a)\downarrow$ then $u_e(a) \geq a$. Also define a computable partial function θ by letting $\theta(a) = (\mu t)[a \in \emptyset'_t]$ if $a \in \emptyset'$ and $\theta(a)\uparrow$ otherwise.

The local strategy for satisfying a single requirement $\mathscr{R}_e$ works as follows. If $\mathscr{R}_e$ receives attention at stage s, this strategy does the following. First it identifies a number

$a \geq e$ that is *not* restrained by strategies of higher priority such that the following conditions holds:

(i) $\Phi^{D_s}_{e,s}(a)\downarrow$
(ii) $u_{e,s}(a) < max(0, \theta_s(a))$

If no such number a exists, the strategy does nothing. Otherwise it puts a restraint on a and *commits* to assigning $f_s(a) = \Phi^{D_s}_{e,s}(a)$. For any such a, this commitment will remain active as long as the strategy has a restraint on this element. Having done all this, the local strategy is declared to be satisfied and will not act again unless either a higher priority puts restraints on a, or the value of $u_{e,s}(a)$ or $\theta_s(a)$ changes. In both cases the strategy gets *injured* and has to reset, releasing all its restraints.

To finish stage s, the global strategy assigns values $f_s(y)$ for all $y \leq s$ as follows: if y is commited to some value assignment of $f_s(y)$ due to a local strategy, then define $f_s(y)$ to be this value. If not, let $f_s(y) = 0$. This finishes the construction and we now turn to the verification.

For each $e, a \in \omega$, let $Z_{e,a} = \{s \in \omega : \mathscr{R}_e$ restrains a at stage $s\}$.

Claim. For each $e, a \in \omega$,

(a) if $\Phi^D_e(a)\uparrow$ then $Z_{e,a}$ is finite;
(b) if $\Phi^D_e(a)\downarrow$ then $Z_{e,a}$ is either finite or cofinite.

Proof. By induction on the priority order. We consider $Z_{e,a}$, assuming that for all $\mathscr{R}_{e'}$ of higher priority, the set $Z_{e',a}$ is either finite or cofinite. First notice that $Z_{e,a} = \emptyset$ if $a < e$ or $a \notin \emptyset'$, so we may assume that $a \geq e$ and $a \in \emptyset'$. If there exists $e' < e$ such that $Z_{e',a}$ is cofinite, then $Z_{e,a}$ is finite because at most one requirement may claim a at a given stage. Suppose that $Z_{e',a}$ is finite for all $e' < e$. Fix t_0 such that for all $e' < e$ and $s \geq t_0$ $\mathscr{R}_{e'}$ does not restrain a at stage s and $\theta_s(a) = \theta(a)$.

Suppose that $\Phi^D_e(a)\uparrow$. Fix $t_1 \geq t_0$ such that $D(b) = D_s(b)$ for all $b \leq \theta(a)$ and all $s \geq t_1$. Then for all $s \geq t_1$, if $\Phi^{D_s}_{e,s}(a)\downarrow$ then we must have $u_{e,s}(a) > \theta(a)$ because otherwise $\Phi^D_e(a)\downarrow$. It follows that for all $s \geq t_1$, requirement $\mathscr{R}_e$ does not restrain a at stage s. Hence $Z_{e,a}$ is finite.

Suppose now that $\Phi^D_e(a)\downarrow$. Fix $t_1 \geq t_0$ such that for all $s \geq t_1$ we have $\Phi^{D_s}_{e,s}(a)\downarrow$ and $D_s(c) = D(c)$ for every $c \leq u_e(a)$. For every $s \geq t_1$, $u_{e,s}(a) = u_{e,t_1}(a)$ and $\theta_s(a) = \theta_{t_1}(a)$ for each $i \leq a$. So properties (i) and (ii) will either hold at each stage $s \geq t_1$, or not hold at each stage $s \geq t_1$. Therefore $Z_{e,a}$ is either finite or cofinite. □

Claim. Each requirement $\mathscr{R}_e$ is satisfied.

Proof. Suppose that Φ^D_e is total for some $e \in \omega$. We will prove that Φ^D_e is not an f-diagonalizing function. Let $A = \left\{a \geq e : (\forall e' < e) Z_{e',a} \text{ is finite}\right\}$. Notice that A is cofinite since for each $e' < e$, there is at most one a such that $Z_{e',a}$ is cofinite.

If for all but finitely many $k \in \omega$, we have $k \in \emptyset' \rightarrow k \in \emptyset'_{u_e(k)}$, then $\emptyset' \leq_T u_e \leq_T D$, contrary to hypothesis. Thus we may let a be the least element of $\{k \in A : k \in \emptyset' \setminus \emptyset'_{u_e(k)}\}$. We then have

(1) $a \geq e$, $\Phi^D_e(a)\downarrow$, $\theta(a) > u_e(a)$
(2) For all $e' < e$, there exists t such that $\mathscr{R}_{e'}$ does not claim a at any stage $s \geq t$.

Therefore we may fix $t \geq a$ such that for all $s \geq t$, we have $\Phi^{D_s}_{e,s}(a)\downarrow$, $\theta_s(a) = \theta(a)$, $u_{e,s}(a) = u_e(a)$, and for each $e' < e$, $R_{e'}$ does not claim a at stage s. Thus, for every $s \geq t$, requirement $\mathscr{R}_e$ claims $a' \leq a$ at stage s. Since $Z_{e,i}$ is either finite or cofinite for each $i \leq a$, it follows that $Z_{e,a}$ is cofinite. By the above argument, we must have $\Phi_e^D(a)\downarrow$, and by construction, $f(a) = \Phi_e^D(a)$. Therefore $\mathscr{R}_e$ is satisfied. □

Claim. The resulting function f_s is Δ^0_2.

Proof. Consider a particular element a. Because of Claim 1, if $e > a$ then $Z_{e,a} = \emptyset$. We have then two cases: Either $Z_{e,a}$ is finite for all $e \leq a$, in which case for all but finitely many s, $f_s(a) = 0$, or $Z_{e,a}$ is cofinite for some e. Then there is a stage s at which requirement $\mathscr{R}_e$ has committed $f_s(a) = \Phi_e^D(a)$ for assignment and has never been injured. Thus f is Δ^0_2. □

□

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