Partial Orders and Immunity in Reverse Mathematics

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Abstract. We identify computability-theoretic properties enabling us to separate various statements about partial orders in reverse mathematics. We obtain simpler proofs of existing separations, and deduce new compound ones. This work is part of a larger program of unification of the separation proofs of various Ramsey-type theorems in reverse mathematics in order to obtain a better understanding of the combinatorics of Ramsey's theorem and its consequences. We also answer a question of Murakami, Yamazaki and Yokoyama about pseudo Ramsey's theorem for pairs.

1 Introduction

Many theorems of "ordinary" mathematics are of the form

$$(\forall X)[\Phi(X) \rightarrow (\exists Y)\Psi(X,Y)]$$

where Φ and Ψ are arithmetic formulas. They can be seen as *mathematical problems*, whose *instances* are sets X such that $\Phi(X)$ holds, and whose *solutions* to X are sets Y such that $\Psi(X,Y)$ holds. For example, König's lemma asserts that every infinite, finitely branching tree admits an infinite path through it.

There exist many ways to calibrate the strength of a mathematical problem. Among them, *reverse mathematics* is a vast foundational program that seeks to determine the weakest axioms necessary to prove ordinary theorems. It uses the framework of subsystems of second-order arithmetic, within the base theory RCA_0, which can be thought of as capturing *computable mathematics*. An *ω-structure* is a structure whose first-order part consists of the standard integers. The ω-models of RCA_0 are those whose second-order part is a *Turing ideal*, that is, a collection of sets $\mathscr{S}$ downward-closed under the Turing reduction and closed under the effective join.

In this setting, a ω-model $\mathscr{M}$ satisfies a mathematical problem P if every P-instance in $\mathscr{M}$ has a solution in $\mathscr{M}$. A standard way of proving that a problem P does not imply another problem Q consists of creating an ω-model $\mathscr{M}$ satisfying P but not Q. Such a model is usually constructed by taking a ground Turing ideal, and extending it by iteratively adding solutions to its P-instances. However, while taking the closure of the collection $\mathscr{M} \cup \{Y\}$ to obtain a Turing ideal, one may add solutions to Q-instances as well. The whole difficulty of this construction consists of finding the right computability-theoretic notion preserved by P but not by Q.

We conduct a program of identification of the computability-theoretic properties enabling us to distinguish various Ramsey-type theorems in reverse mathematics, but also under computable and Weihrauch reducibilities. This program puts emphasis on the interplay between computability theory and reverse mathematics, the former providing tools to separate theorems in reverse mathematics over standard models, and the latter exhibiting new computability-theoretic properties.

Among the theorems studied in reverse mathematics, the ones coming from Ramsey's theory play a central role. Their strength are notoriously hard to gauge, and required the development of involved computability-theoretic frameworks. Perhaps the most well-known example is Ramsey's theorem.

Definition 1 (Ramsey's theorem). *A subset H of ω is* homogeneous *for a coloring $f : [\omega]^n \to k$ (or f-*homogeneous*) if each n-tuples over H are given the same color by f. RT^n_k is the statement "Every coloring $f : [\omega]^n \to k$ has an infinite f-homogeneous set".*

Jockusch [11] conducted a computational analysis of Ramsey's theorem. He proved in particular that RT^n_k implied the existence of the halting set whenever $n \geq 3$. There has been a lot of literature around the strength of Ramsey's theorem for pairs [4,6,9,19] and its consequences [3,5,10]. We focus on some mathematical statements about partial orders which are consequences of Ramsey's theorem for pairs.

Definition 2 (Chain-antichain). *A* chain *in a partial order $(P, \leq_P)$ is a set $S \subseteq P$ such that $(\forall x, y \in S)(x \leq_P y \vee y \leq_P x)$. An* antichain *in P is a set $S \subseteq P$ such that $(\forall x, y \in S)(x \neq y \to x|_P y)$ (where $x|_P y$ means that $x \not\leq_P y \wedge y \not\leq_P x$). CAC is the statement "every infinite partial order has an infinite chain or an infinite antichain."*

The chain-antichain principle was introduced by Hirschfeldt and Shore [10] together with the ascending descending sequence (ADS). They studied extensively cohesive and stable versions of the statements, and proved that CAC is computationally weak, in that it does not even imply the existence of a diagonally non-computable function. However, their proof has an ad-hoc flavor, in that it is a direct separation involving the two statements. Later, Lerman, Solomon and Towsner [13] separated ADS from CAC over ω-models by using an involved iterated forcing argument.

In this paper, we revisit the two proofs and emphasis on the combinatorial nature of the principles by identifying the computability-theoretic properties separating them. Those properties happen to be very natural and coincide on co-c.e. sets to some well-known computability-theoretic notions, namely, immunity and hyperimmunity. The proof of the separation of ADS from CAC is significantly simpler and more modular, as advocated by the author in [16].

1.1 Notation and definitions

Given two sets A and B, we denote by $A < B$ the formula $(\forall x \in A)(\forall y \in B)[x < y]$ and by $A \subseteq^* B$ the formula $(\forall^\infty x \in A)[x \in B]$, meaning that A is included in B *up to finitely many elements*. A *Mathias condition* is a pair (F, X) where F is a finite set, X is an infinite set and $F < X$. A condition (F_1, X_1) *extends* (F, X) (written $(F_1, X_1) \leq (F, X)$) if $F \subseteq F_1$, $X_1 \subseteq X$ and $F_1 \setminus F \subset X$. A set G *satisfies* a Mathias condition (F, X) if $F \subset G$ and $G \setminus F \subseteq X$.

2 Preservation of Properties for Co-c.e. Sets

Ramsey's theorem for k colors has a deeply disjunctive nature. One cannot know in a finite amount of time whether a coloring will admit an infinite homogeneous set for a fixed color, and one must therefore build multiple homogeneous sets simultaneously, namely, one for each color. This disjunction was exploited by the author to show for example that ADS does not preserve 2 hyperimmunities simultaneously, whereas the Erdős-Moser theorem does [16]. This idea was also used in the context of computable reducibility to show that RT^2_{k+1} does not computably reduce to RT^2_k whenever $k \geq 1$, by showing that RT^2_k preserves 2 among $k+1$ hyperimmunities simultaneously whereas RT^2_{k+1} does not [18]. In this section, we shall see that this disjunctive flavor disappears whenever considering co-c.e. sets. In particular, RT^2_2 admits preservation of countably many hyperimmune co-c.e. sets simultaneously.

Definition 3 (Hyperimmunity). *An* array *is a sequence of mutually disjoint finitely coded sets. A set A is X-*hyperimmune *if for every X-c.e. array $F_0, F_1, \dots$, there is some i such that $F_i \cap A = \emptyset$.*

Equivalently, a set is X-hyperimmune if its principal function is not dominated by any X-computable function, where the *principal function* p_A of a set $A = \{x_0 < x_1 < \dots\}$ is defined by $p_A(i) = x_i$.

Definition 4 (Preservation of hyperimmunity for co-c.e. sets). *A Π^1_2 statement* P *admits* preservation of hyperimmunity for co-c.e. sets *if for every set Z, every sequence of Z*-co-c.e. *Z-hyperimmune sets $A_0, A_1, \dots$ and every P-instance $X \leq_T Z$, there is a solution Y to X such that the A's are $Y \oplus Z$-hyperimmune.*

Hirschfeldt and Shore [10] proved that CAC is equivalent to the existence of homogeneous sets for semi-transitive colorings. A coloring $f : [\mathbb{N}]^2 \to 2$ is *semi-transitive* if whenever $f(x,y) = 1$ and $f(y,z) = 1$, then $f(x,z) = 1$ for $x < y < z$.

Theorem 5. CAC *admits preservation of hyperimmunity for co-c.e. sets.*

Proof. Fix a set Z and a countable sequence of Z-co-c.e. Z-hyperimmune sets $A_0, A_1, \dots$ Let $f : [\omega]^2 \to 2$ be a Z-computable semi-transitive coloring. We shall assume that there is no infinite Z-computable f-homogeneous set for color 0, otherwise we are done. We will build an infinite set G f-homogeneous for color 1 such that the A's are $G \oplus Z$-hyperimmune. The construction is done by a Mathias forcing (F, X), where F is a finite set, X is an infinite Z-computable set such that $max(F) < min(X)$, and for every $x \in X$, $F \cup \{x\}$ is f-homogeneous for color 1. The condition extension is the usual Mathias extension. A set G *satisfies* (F, X) if it satisfies the Mathias condition (F, X) and is f-homogeneous for color 1. Lemma 6 shows that every sufficiently generic filter for this notion of forcing yields an infinite set.

Lemma 6. *Every condition $c = (F, X)$ has an extension (E, Y) such that $|E| > |F|$.*

In what follows, we say that a condition c *forces* a formula property $\varphi(G)$ if $\varphi(G)$ holds for every set G satisfying c.

Lemma 7. *For every condition $c = (F, X)$ and every pair of indices e, i, there is an extension forcing $\Phi_e^{G \oplus Z}$ not to dominate p_{A_i}.*

Proof. Define the Z-partial computable function h which on input x, searches for a finite set $E_x \subseteq X$ f-homogeneous for color 1 such that $\Phi_e^{(F \cup E_x) \oplus Z}(x) \downarrow$. If found, $h(x) = \Phi_e^{(F \cup E_x) \oplus Z}(x)$, otherwise $h(x) \uparrow$. We have two cases.

- Case 1: h is total. By Z-hyperimmunity of p_{A_i}, there are infinitely many x such that $h(x) < p_{A_i}(x)$. If there is such an x such that the set $Y = \{y \in X : (\forall z \in E_x) f(z, y) = 1\}$ is infinite, then the condition $(F \cup E_x, Y)$ is an extension of c forcing $\Phi_e^{G \oplus Z}(x) < p_{A_i}(x)$. If there is no such x, then by semi-transitivity of f, for every x such that $h(x) < p_{A_i}(x)$, for almost every $y \in X$, $f(\max(E_x), y) = 0$. Since A_i is co-c.e., one can find a Z-computable infinite subset Y of $\{\max(E_x) : h(x) < p_{A_i}(x)\}$. The set Y is Z-computable and limit-homogeneous for color 0, and therefore computes an infinite f-homogeneous set for color 0, contradicting our assumption.
- Case 2: there is some x such that $h(x) \uparrow$. By definition of h, the condition c already forces $\Phi_e^{G \oplus Z}(x) \uparrow$. ☐

Corollary 8. RT^2_2 *admits preservation of hyperimmunity for co-c.e. sets.*

Proof. Bovykin and Weiermann [2] studied the reverse mathematics of the Erdős-Moser theorem (EM) and proved that $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \leftrightarrow [\mathsf{CAC} \wedge \mathsf{EM}]$. The author proved in [16] that EM admits preservation of hyperimmunity. Together with Theorem 5, we deduce that $\mathsf{CAC} \wedge \mathsf{EM}$, hence RT^2_2, admits preservation of hyperimmunity for co-c.e. sets.

3 CAC and Constant-Bound Immunity

Hirschfeldt and Shore [10] separated CAC from DNC in reverse mathematics by a direct construction. DNC is the statement asserting, for every set X, the existence of a function f such that $f(e) \neq \Phi_e^X(e)$ for every e. In this section, we extract the core of the combinatorics of their forcing argument to exhibit a computability-theoretic property separating the two notions, namely, constant-bound immunity.

Definition 9 (Constant-bound immunity). *A* k-enumeration *(k-enum) of a set A is an infinite sequence of k-sets $F_0 < F_1 < \dots$ such that for every $i \in \omega$, $F_i \cap A \neq \emptyset$. A* constant-bound enumeration *(c.b-enum) of a set A is a k-enumeration of A for some $k \in \omega$. A set A is* k-immune (c.b-immune) relative to *X if it admits no X-computable k-enumeration (c.b-enumeration).*

In particular, 1-immunity coincides with the standard notion of immunity. Also note that one can easily create a c.b-immune set computing no effectively immune set. The following lemma shows that c.b-immunity and immunity coincide for co-c.e. sets.

Lemma 10. *An X-co-c.e. set A is c.b-immune relative to X iff it is X-immune.*

Definition 11 (Preservation of c.b-immunity). *A Π^1_2 statement P admits preservation of c.b-immunity if for every set Z, every set A which is c.b-immune relative to X, and every P-instance $X \leq_T Z$, there is a solution Y to X such that A is c.b-immune relative to $Y \oplus Z$.*

We can easily relate the notion of preservation of c.b-immunity with the existing notion of constant-bound enumeration avoidance defined by Liu [14] to separate RT^2_2 from WWKL over RCA_0.

Lemma 12. *If P admits preservation of c.b-immunity, then it admits constant-bound enumeration avoidance.*

Theorem 13. CAC *admits preservation of c.b-immunity.*

Proof. Let A be a set c.b-immune relative to some set Z, and let $f : [\omega]^2 \to 2$ be a Z-computable semi-transitive coloring. Assume that there is no infinite f-homogeneous set H such that A is c.b-immune relative to $H \oplus Z$, otherwise we are done. We will build two infinite sets G_0 and G_1, such that G_i is f-homogeneous for color i for each $i < 2$, and such that A is c.b-immune relative to $G_i \oplus Z$ for some $i < 2$.

The construction is done by a variant of Mathias forcing (F_0, F_1, X), where F_0 and F_1 are finite sets, X is infinite set such that $max(F_0, F_1) < min(X)$, and A is c.b-immune relative to $X \oplus Z$. Moreover, we require that for every $i < 2$ and every $x \in X$, $F_i \cup \{x\}$ is f-homogeneous for color i. A condition (E_0, E_1, Y) *extends* (F_0, F_1, X) if (E_i, Y) Mathias extends (F_i, X) for each $i < 2$. A pair of sets G_0, G_1 *satisfies* a condition $c = (F_0, F_1, X)$ if G_i is f-homogeneous for color i and satisfies the Mathias condition (F_i, X) for each $i < 2$.

Lemma 14. *For every condition $c = (F_0, F_1, X)$ and every $i < 2$, there is an extension (E_0, E_1, Y) of c such that $|E_i| > |F_i|$.*

In what follows, we interpret $\Phi_0, \Phi_1, \dots$ as Turing functionals outputting non-empty finite sets such that if $\Phi^X_e(x)$ and $\Phi^X_e(x+1)$ both halt, $max(\Phi^X_e(x)) < min(\Phi^X_e(x+1))$. We want to satisfy the following requirements for each $e_0, k_0, e_1, k_1 \in \omega$:

$$\mathscr{R}_{e_0,k_0,e_1,k_1} : \quad \mathscr{R}^{G_0}_{e_0,k_0} \quad \vee \quad \mathscr{R}^{G_1}_{e_1,k_1}$$

where $\mathscr{R}^G_{e,k}$ is the requirement

$$(\exists x)\left(\Phi^{G\oplus Z}_e(x)\uparrow \vee |\Phi^{G\oplus Z}_e(x)| > k \vee \Phi^{G\oplus Z}_e(x) \cap A = \emptyset\right)$$

In other words, $\mathscr{R}^G_{e,k}$ asserts that $\Phi^{G\oplus Z}_e$ is not a k-enumeration of A. A condition c *forces* a formula $\varphi(G_0, G_1)$ if $\varphi(G_0, G_1)$ holds for every pair of *infinite* sets G_0, G_1 satisfying c.

Lemma 15. *For every condition c and every vector of indices $e_0, k_0, e_1, k_1 \in \omega$, there is an extension d of c forcing $\mathscr{R}_{e_0,k_0,e_1,k_1}$.*

Proof. Fix a condition $c = (F_0, F_1, X)$, and let $P_0, P_1, \dots$ be an $X \oplus Z$-computable sequence of sets where $P_n = \Phi_{e_0}^{(F_0 \cup E_0)\oplus Z}(x_0) \cup \Phi_{e_1}^{(F_1 \cup E_1)\oplus Z}(x_1)$ for a pair of sets $E_1 < E_0 \subseteq X$ and some $x_0, x_1 \in \omega$ such that E_0 is f-homogeneous for color 0, $E_1 \cup \{y\}$ is f-homogeneous for color 1 for each $y \in E_0$, and for each $i < 2$, $\max(P_{n-1}) < \min(\Phi_{e_i}^{(F_i \cup E_i)\oplus Z}(x_i))$ and $|\Phi_{e_i}^{(F_i \cup E_i)\oplus Z}(x_i)| \leq k_i$. We have two cases.

- Case 1: the sequence of the P's is finite and is defined, say to level $n-1$. If there is a pair of infinite sets G_0, G_1 satisfying c and some $x_1 \in \omega$ such that $\Phi_{e_1}^{G_1 \oplus Z}(x_1)\downarrow$, $\max(P_{n-1}) < \min(\Phi_{e_1}^{G_1 \oplus Z}(x_1))$, and $|\Phi_{e_1}^{G_1 \oplus Z}(x_1)| \leq k_1$, then let $E_1 \subseteq G_1$ be such that $F_1 \cup E_1$ is an initial segment of G_1 for which $\Phi_{e_1}^{(F_1 \cup E_1)\oplus Z}(x_1)\downarrow$. The set $Y = \{y \in X : E_1 \cup \{y\}$ is f-homogeneous for color 1 $\}$ is a superset of G_1, hence is infinite. The condition $d = (F_0, F_1 \cup E_1, Y)$ is an extension of c forcing $\mathscr{R}_{e_0,k_0}^{G_0}$, hence forcing $\mathscr{R}_{e_0,k_0,e_1,k_1}$. If there is no such pair of infinite sets G_0, G_1, then the condition c already forces $\mathscr{R}_{e_1,k_1}^{G_1}$, hence $\mathscr{R}_{e_0,k_0,e_1,k_1}$.
- Case 2: the sequence of the P's is infinite. By c.b-immunity of A relative to $X \oplus Z$, $P_n \cap A = \emptyset$ for some $n \in \omega$. Let $E_1 < E_0 \subseteq X$ and $x_0, x_1 \in \omega$ witness the existence of P_n. If $Y_0 = \{y \in X : E_0 \cup \{y\}$ is f-homogeneous for color 1 $\}$ is infinite, then the condition $(F_0 \cup E_0, F_1, Y_0)$ is an extension of c forcing $\mathscr{R}_{e_0,k_0}^{G_0}$. If Y_0 is finite, then for almost every $y \in X$, there is some $x_y \in E_0$ such that $f(x_y, y) = 1$, and by transitivity of f for color 1, $E_1 \cup \{y\}$ is f-homogeneous for color 1. Indeed, E_1 is f-homogeneous for color 1 and for each $x \in E_1$, $f(x, x_y) = f(x_y, y) = 1$. In this case, $(F_0, F_1 \cup E_1, Y_1)$ is an extension of c forcing $\mathscr{R}_{e_1,k_1}^{G_1}$, for some $Y_1 =^* X$. In both cases, there is an extension of c forcing $\mathscr{R}_{e_0,k_0,e_1,k_1}$.

This completes the proof of Theorem 13. ◻

Theorem 16. DNC *does not admit preservation of c.b-immunity.*

Proof (Proof sketch). Let $\mu_{\emptyset'}$ be the modulus function of $\emptyset'$, that is, such that $\mu_{\emptyset'}(x)$ is the minimum stage s at which $\emptyset'_s \upharpoonright x = \emptyset' \upharpoonright x$.

Computably split ω into countably many columns $X_0, X_1, \dots$ of infinite size. For example, set $X_i = \{\langle i, n\rangle : n \in \omega\}$ where $\langle \cdot, \cdot \rangle$ is a bijective function from ω^2 to ω. For each i, let F_i be the set of the $\mu_{\emptyset'}(i)$ first elements of X_i. The sequence $F_0, F_1, \dots$ is $\emptyset'$-computable. By a simple finite injury priority argument (see appendix), one can construct a c.e. set W such that the Δ^0_2 set $A = \bigcup_i F_i \setminus W$ is c.b-immune, and such that $|X_i \cap W| \leq i$. We claim that every DNC function computes an infinite subset of A.

Let f be any DNC function. By a classical theorem about DNC functions (see Bienvenu et al. [1] for a proof), f computes a function $g(\cdot, \cdot, \cdot)$ such that whenever $|W_e| \leq n$, then $g(e, n, i) \in X_i \setminus W_e$. For each i, let e_i be the index of the c.e. set $W_{e_i} = W \cap X_i$, and let $n_i = g(e_i, i, i)$. Since $|X_i \cap W| \leq i$, $|W_{e_i}| \leq i$, hence $n_i = g(e_i, i, i) \in X_i \setminus W_{e_i} = X_i \setminus W$. We then have two cases.

- Case 1: $n_i \in F_i$ for infinitely many i's. One can f-computably find infinitely many of them since $\mu_{\emptyset'}$ is left-c.e. and the sequence of the n's is f-computable. Therefore, one can f-computably find an infinite subset of $\bigcup_i F_i \setminus W = A$.

– Case 2: $n_i \in F_i$ for only finitely many i's. Then the sequence of the n_i's dominates the modulus function $\mu_{\emptyset'}$, and therefore computes the halting set. Since the set A is Δ^0_2, f computes an infinite subset of A. ☐

Corollary 17 (Hirschfeldt and Shore [10]). $\mathsf{RCA}_0 \wedge \mathsf{CAC} \nvdash \mathsf{DNC}$.

4 ADS and Dependent Hyperimmunity

Lerman, Solomon and Towsner [13] separated the ascending descending sequence principle from a stable version of CAC by using a very involved iterated forcing argument. According to our previous simplification of their general framework [16], we reformulate their proof in terms of preservation of dependent hyperimmunity, and extend it to pseudo Ramsey's theorem for pairs.

Definition 18 (Ascending descending sequence). *Given a linear order $(L, <_L)$, an* ascending *(*descending*) sequence is a set S such that for every $x <_{\mathbb{N}} y \in S$, $x <_L y$ $(x >_L y)$.* ADS *is the statement "Every infinite linear order admits an infinite ascending or descending sequence".*

Pseudo Ramsey's theorem for pairs was introduced by Friedman [7] and later studied by Friedman and Pelupessy [8], and Murakami, Yamazaki and Yokoyama in [15] who proved that it is between the chain antichain principle and the ascending descending sequence principle over RCA_0. Steila [20] and the author [17] independently proved that it is actually equivalent to ADS.

Definition 19 (Pseudo Ramsey's theorem). *A set $H = \{x_0 < x_1 < \dots\}$ is* pseudo-homogeneous *for a coloring $f : [\mathbb{N}]^n \to k$ if $f(x_i, \dots, x_{i+n-1}) = f(x_j, \dots, x_{j+n-1})$ for every $i, j \in \mathbb{N}$. psRT^n_k is the statement "Every coloring $f : [\mathbb{N}]^n \to k$ has an infinite pseudo-homogeneous set".*

Definition 20 (Dependent hyperimmunity). *A formula $\varphi(U, V)$ is* essential *if for every $x \in \omega$, there is a finite set $R > x$ such that for every $y \in \omega$, there is a finite set $S > y$ such that $\varphi(R, S)$ holds. A pair of sets $A_0, A_1 \subseteq \omega$ is* dependently X-hyperimmune *if for every essential $\Sigma^{0,X}_1$ formula $\varphi(U, V)$, $\varphi(R, S)$ holds for some $R \subseteq \overline{A}_0$ and $S \subseteq \overline{A}_1$.*

In particular, if the pair A_0, A_1 is dependently hyperimmune, then A_0 and A_1 are both hyperimmune.

Definition 21 (Preservation of dependent hyperimmunity). *A Π^1_2 statement* P *admits* preservation of dependent hyperimmunity *if for every set Z, every pair of dependently Z-hyperimmune sets $A_0, A_1 \subseteq \omega$ and every* P*-instance $X \leq_T Z$, there is a solution Y to X such that A_0, A_1 are dependently $Y \oplus Z$-hyperimmune.*

A partial order $(P, \leq_P)$ is *stable* if either $(\forall i \in P)(\exists s)[(\forall j > s)(j \in P \to i \leq_P j) \vee (\forall j > s)(j \in P \to i \mid_P j)]$ or $(\forall i \in P)(\exists s)[(\forall j > s)(j \in P \to i \geq_P j) \vee (\forall j > s)(j \in P \to i \mid_P j)]$. SCAC is the restriction of CAC to stable partial orders. A simple finite injury priority argument shows that SCAC does not admit preservation of dependent hyperimmunity.

Theorem 22. *There exists a computable, stable semi-transitive coloring $f : [\omega]^2 \to 2$ such that the pair $\overline{A}_0, \overline{A}_1$ is dep. hyperimmune, where $A_i = \{x : \lim_s f(x,s) = i\}$.*

Corollary 23. SCAC *does not admit preservation of dependent hyperimmunity.*

Proof. Let $f : [\omega]^2 \to 2$ be the coloring of Theorem 22. By construction, the pair $\overline{A}_0$, $\overline{A}_1$ is dependently hyperimmune, where $A_i = \{x : \lim_s f(x,s) = i\}$. Let H be an infinite f-homogeneous set. In particular, $H \subseteq A_0$ or $H \subseteq A_1$. We claim that the pair $\overline{A}_0, \overline{A}_1$ is not dependently H-hyperimmune. The $\Sigma^{0,H}_1$ formula $\varphi(U,V)$ defined by $U \neq \emptyset \wedge V \neq \emptyset \wedge U \cup V \subseteq H$ is essential since H is infinite. However, if there is some $R \subseteq A_1$ and $S \subseteq A_0$ such that $\varphi(R,S)$ holds, then $H \cap A_0 \neq \emptyset$ and $H \cap A_1 \neq \emptyset$, contradicting the choice of H. Therefore $\overline{A}_0, \overline{A}_1$ is not dependently H-hyperimmune. Hirschfeldt and Shore [10] proved that SCAC is equivalent to stable semi-transitive Ramsey's theorem for pairs over RCA_0. Therefore SCAC does not admit preservation of dependent hyperimmunity. ☐

We will now prove the positive preservation result.

Theorem 24. *For every $k \geq 2$, psRT^2_k admits preservation of dep. hyperimmunity.*

Proof. The proof is done by induction over $k \geq 2$. Fix a pair of sets $A_0, A_1 \subseteq \omega$ dependently Z-hyperimmune for some set Z. Let $f : [\omega]^2 \to k$ be a Z-computable coloring and suppose that there is no infinite set H over which f avoids at least one color, and such that the pair A_0, A_1 is dependently $H \oplus Z$-hyperimmune, as otherwise, we are done by induction hypothesis. We will build k infinite sets $G_0, \dots, G_{k-1}$ such that G_i is pseudo-homogeneous for f with color i for each $i < k$ and such that A_0, A_1 is dependently $G_i \oplus Z$-hyperimmune for some $i < k$. The sets $G_0, \dots, G_{k-1}$ are built by a variant of Mathias forcing $(F_0, \dots, F_{k-1}, X)$ such that

(i) $F_i \cup \{x\}$ is pseudo-homogeneous for f with color i for each $x \in X$
(ii) X is an infinite set such that A_0, A_1 is dependently $X \oplus Z$-hyperimmune

A condition $d = (H_0, \dots, H_{k-1}, Y)$ *extends* $c = (F_0, \dots, F_{k-1}, X)$ (written $d \leq c$) if (H_i, Y) Mathias extends (F_i, X) for each $i < k$. A tuple of sets $G_0, \dots, G_{k-1}$ *satisfies* c if for every $n \in \omega$, there is an extension $d = (H_0, \dots, H_{k-1}, Y)$ of c such that $G_i \upharpoonright n \subseteq H_i$ for each $i < k$. Informally, $G_0, \dots, G_{k-1}$ satisfy c if the sets are generated by a decreasing sequence of conditions extending c. In particular, G_i is pseudo-homogeneous for f with color i and satisfies the Mathias condition (F_i, X). The first lemma shows that every sufficiently generic filter yields a k-tuple of infinite sets.

Lemma 25. *For every condition $c = (F_0, \dots, F_{k-1}, X)$ and every $i < k$, there is an extension $d = (H_0, \dots, H_{k-1}, Y)$ of c such that $|H_i| > |F_i|$.*

Fix an enumeration $\varphi_0(G,U,V), \varphi_1(G,U,V), \dots$ of all $\Sigma^{0,Z}_1$ formulas. We want to satisfy the following requirements for each $e_0, \dots, e_{k-1} \in \omega$:

$$\mathscr{R}_{\vec{e}} : \quad \mathscr{R}^{G_0}_{e_0} \quad \vee \quad \dots \quad \vee \quad \mathscr{R}^{G_{k-1}}_{e_{k-1}}$$

where $\mathscr{R}^G_e$ is the requirement "$\varphi_e(G,U,V)$ essential $\to \varphi_e(G,R,S)$ for some $R \subseteq \overline{A}_0$ and $S \subseteq \overline{A}_1$". We say that a condition c *forces* $\mathscr{R}_{\vec{e}}$ if $\mathscr{R}_{\vec{e}}$ holds for every k-tuple of sets satisfying c. Note that the notion of satisfaction has a precise meaning given above.

Lemma 26. *For every condition c and every k-tuple of indices $e_0, \dots, e_{k-1} \in \omega$, there is an extension d of c forcing $\mathscr{R}_{\vec{e}}$.*

Proof. Fix a condition $c = (F_0, \dots, F_{k-1}, X)$. Let $\psi(U, V)$ be the $\Sigma_1^{0,X \oplus Z}$ formula which holds if there is a k-tuple of sets $E_0, \dots, E_{k-1} \subseteq X$ and a $z \in X$ such that for each $i < k$,

(i) $z > max(E_i)$
(ii) $F_i \cup E_i \cup \{z\}$ is pseudo-homogeneous for color i.
(iii) $\varphi_{e_i}(F_i \cup E_i, U_i, V_i)$ holds for some $U_i \subseteq U$ and $V_i \subseteq V$

Suppose that c does not force $\mathscr{R}_{\vec{e}}$, otherwise we are done.

We claim that ψ is essential. Since c does not force $\mathscr{R}_{\vec{e}}$, there is a k-tuple of infinite sets $G_0, \dots, G_{k-1}$ satisfying c and such that $\varphi_{e_i}(G_i, U, V)$ is essential for each $i < k$. Fix some $x \in \omega$. By definition of being essential, there are some finite sets $R_0, \dots, R_{k-1} > x$ such that for every $y \in \omega$, there are finite sets $S_0, \dots, S_{k-1} > y$ such that $\varphi_{e_i}(G_i, R_i, S_i)$ holds for each $i < k$. Let $R = \bigcup R_i$ and fix some $y \in \omega$. There are finite sets $S_0, \dots, S_{k-1} > y$ such that $\varphi_{e_i}(G_i, R_i, S_i)$ holds for each $i < k$. Let $S = \bigcup S_i$. By continuity, there are finite sets $E_0, \dots, E_{k-1}$ such that $G_i \upharpoonright max(E_i) = F_i \cup E_i$ and $\varphi_{e_i}(F_i \cup E_i, R_i, S_i)$ holds for each $i < k$. By our precise definition of satisfaction, we can even assume without loss of generality that $(F_0 \cup E_0, \dots, F_{k-1} \cup E_{k-1}, Y)$ is a valid extension of c for some infinite set $Y \subseteq X$. Let $z \in Y$. In particular, by the definition of being a condition extending c, $z \in X$, $z > max(E_0, \dots, E_{k-1})$ and $F_i \cup E_i \cup \{z\}$ is pseudo-homogeneous for color i for each $i < k$. Therefore $\psi(R, S)$ holds, as witnessed by $E_0, \dots, E_{k-1}$ and z. Thus $\psi(R, S)$ is essential.

Since A_0, A_1 is dependently $X \oplus Z$-hyperimmune, then $\psi(R, S)$ holds for some $R \subseteq \overline{A}_0$ and some $S \subseteq \overline{A}_1$. Let $E_0, \dots, E_{k-1} \subseteq X$ be the k-tuple of sets and $z \in X$ be the integer witnessing $\psi(R, S)$. Let $i < k$ be such that the set $Y = \{w \in X \setminus [0, max(E_i)] : f(z, w) = i\}$ is infinite. The condition $d = (F_0, \dots, F_{i-1}, F_i \cup E_i \cup \{z\}, F_{i+1}, \dots, F_{k-1}, Y)$ is a valid extension of c forcing $\mathscr{R}_{\vec{e}}$. ☐

Theorem 27. *Fix some set Z and a pair of sets A_0, A_1 dependently Z-hyperimmune. If Y is sufficiently random relative to Z, then the pair A_0, A_1 is dependently $Y \oplus Z$-hyperimmune.*

Corollary 28. WWKL *admits preservation of dependent hyperimmunity.*

Proof. Let Z be a set and A_0, A_1 be a pair of dependently Z-hyperimmune sets. Fix a Z-computable tree of positive measure $T \subseteq 2^{<\omega}$. By Theorem 27, the pair A_0, A_1 is dependently $Y \oplus Z$-hyperimmune for some Martin-Löf random Y relative to Z. By Kučera [12], Y is, up to finite prefix, a path through T. ☐

Corollary 29. *For every $k \geq 2$,* $\mathsf{RCA}_0 \wedge \mathsf{psRT}^2_k \wedge \mathsf{WWKL} \nvdash \mathsf{SCAC}$.

Whenever requiring the sets A_0 and A_1 to be co-c.e., we recover the standard notion of hyperimmunity. Therefore, the restriction of the preservation of dependent hyperimmunity to co-c.e. sets is not a good computability-theoretic property to distinguish consequences of Ramsey's theorem for pairs.

Lemma 30. *Fix two sets A_0, A_1 such that A_0 is X-co-c.e. The pair A_0, A_1 is dependently X-hyperimmune iff A_0 and A_1 are X-hyperimmune.*

Corollary 31. RT^2_2 *admits preservation of dependent hyperimmunity for co-c.e. sets.*

References

1. Bienvenu, L., Patey, L., Shafer, P.: On the logical strengths of partial solutions to mathematical problems (2015), submitted. Available at `http://arxiv.org/abs/1411.5874`
2. Bovykin, A., Weiermann, A.: The strength of infinitary Ramseyan principles can be accessed by their densities. Annals of Pure and Applied Logic p. 4 (2005), to appear.
3. Cholak, P.A., Giusto, M., Hirst, J.L., Jockusch Jr, C.G.: Free sets and reverse mathematics. Reverse mathematics 21, 104–119 (2001)
4. Cholak, P.A., Jockusch, C.G., Slaman, T.A.: On the strength of Ramsey's theorem for pairs. Journal of Symbolic Logic 66(01), 1–55 (2001)
5. Csima, B.F., Mileti, J.R.: The strength of the rainbow Ramsey theorem. Journal of Symbolic Logic 74(04), 1310–1324 (2009)
6. Dzhafarov, D.D., Jockusch, C.G.: Ramsey's theorem and cone avoidance. Journal of Symbolic Logic 74(2), 557–578 (2009)
7. Friedman, H.: Adjacent ramsey theory. preprint, https://u.osu.edu/friedman.8/ 8 (2010)
8. Friedman, H., Pelupessy, F.: Independence of Ramsey theorem variants using ε_0. Proc. Amer. Math. Soc. 144(2), 853–860 (2016)
9. Hirschfeldt, D.R., Jockusch, C.G., Kjos-Hanssen, B., Lempp, S., Slaman, T.A.: The strength of some combinatorial principles related to Ramsey's theorem for pairs. Computational Prospects of Infinity, Part II: Presented Talks, World Scientific Press, Singapore pp. 143–161 (2008)
10. Hirschfeldt, D.R., Shore, R.A.: Combinatorial principles weaker than Ramsey's theorem for pairs. Journal of Symbolic Logic 72(1), 171–206 (2007)
11. Jockusch, C.G.: Ramsey's theorem and recursion theory. Journal of Symbolic Logic 37(2), 268–280 (1972)
12. Kučera, A.: Measure, Π^0_1 classes, and complete extensions of PA. Lecture Notes in Mathematics 1141, 245–259 (1985)
13. Lerman, M., Solomon, R., Towsner, H.: Separating principles below Ramsey's theorem for pairs. Journal of Mathematical Logic 13(02), 1350007 (2013)
14. Liu, L.: Cone avoiding closed sets. Transactions of the American Mathematical Society 367(3), 1609–1630 (2015), `http://dx.doi.org/10.1090/S0002-9947-2014-06049-2`
15. Murakami, S., Yamazaki, T., Yokoyama, K.: On the ramseyan factorization theorem. In: Language, Life, Limits, pp. 324–332. Springer (2014)
16. Patey, L.: Iterative forcing and hyperimmunity in reverse mathematics. Computability (2015), to appear.
17. Patey, L.: The reverse mathematics of Ramsey-type theorems. Ph.D. thesis, Université Paris Diderot (2016)
18. Patey, L.: The weakness of being cohesive, thin or free in reverse mathematics. Israel Journal of Mathematics (2016), to appear. Available at `http://arxiv.org/abs/1502.03709`
19. Seetapun, D., Slaman, T.A.: On the strength of Ramsey's theorem. Notre Dame Journal of Formal Logic 36(4), 570–582 (1995)
20. Steila, S., Yokoyama, K.: Reverse mathematical bounds for the termination theorem, to appear

A Preservation of Properties for Co-c.e. Sets

Proof. Take any $x \in X$ such that the set $Y = \{y \in X : f(x, y) = 1\}$ is infinite. Such an x must exist, otherwise the set X is limit-homogeneous for color 0 and one can X-compute, hence Z-compute, an infinite f-homogeneous set for color 0, contradicting our hypothesis. Take $(F \cup \{x\}, Y)$ as the desired extension.

B CAC and Constant-Bound Immunity

Proof (Proof of Lemma 12). Fix a non-empty class $\mathscr{C} \subseteq 2^\omega$, and let $A = \{\sigma : \mathscr{C} \cap [\sigma] \neq \emptyset\}$. We claim that the degrees of the c.b-enums of A and of $\mathscr{C}$ coincide. Any c.b-enum of $\mathscr{C}$ is a c.b-enum of A. Conversely, let $F_0 < F_1 < \dots$ be a c.b-enum of $\mathscr{C}$. We can computably thin it out and normalize it into an enumeration $E_0 < E_1 < \dots$ such that $|\sigma| = i$ for every $\sigma \in E_i$. □

Proof (Proof of Lemma 10). We first prove that if A is not X-immune, then it is not c.b-immune relative to X. Let W be an infinite X-computable infinite subset of A. Let $\varphi(U)$ be the $\Sigma^{0,X}$ formula which holds if $U \cap W \neq \emptyset$. The formula $\varphi(U)$ is essential, but there is no set $R \subseteq \overline{A}$ such that $\varphi(R)$ holds. Therefore, A is not c.b-immune relative to X.

We now show by induction over k if A is X-co-c.e. and has an X-computable k-enumeration $F_0, F_1, \dots$ then it has an infinite X-computable subset. If $k = 1$, then it is already an infinite subset of A. Suppose now that $k \geq 2$. If there are infinitely many $i \in \omega$ such that $min(F_i) \in \overline{A} \neq \emptyset$, then since A is X-co-c.e., one can find an X-computable infinite set S of such i's. The sequence $\{F_i \setminus min(F_i) : i \in S\}$ is an X-computable $(k-1)$-enumeration of A, and by induction hypothesis, there is an X-computable subset of A. If there are only finitely many such i's, then the sequence $\{min(F_i) : i \in \omega)\}$ is, up to finite changes, an infinite X-computable subset of X. □

Proof. Take any $x \in X$ such that the set $Y = \{y \in X : f(x, y) = i\}$ is infinite. Such an x must exist, otherwise the set X is limit-homogeneous for color $1-i$ and one can X-compute an infinite f-homogeneous set, contradicting our hypothesis. Let $E_i = F_i \cup \{x\}$ and $E_{1-i} = F_{1-i}$, and take (E_0, E_1, Y) as the desired extension.

Proof (Proof of Theorem 16). Let $\mu_{\emptyset'}$ be the modulus function of $\emptyset'$, that is, such that $\mu_{\emptyset'}(x)$ is the minimum stage s at which $\emptyset'_s \upharpoonright x = \emptyset' \upharpoonright x$. The sketch of the proof is the following:

Computably split ω into countably many columns $X_0, X_1, \dots$ of infinite size. For example, set $X_i = \{\langle i, n\rangle : n \in \omega\}$ where $\langle \cdot, \cdot \rangle$ is a bijective function from ω^2 to ω. For each i, let F_i be the set of the $\mu_{\emptyset'}(i)$ first elements of X_i. The sequence $F_0, F_1, \dots$ is $\emptyset'$-computable. Assume for now that we have defined a c.e. set W such that the Δ^0_2 set $A = \bigcup_i F_i \setminus W$ is c.b-immune, and such that $|X_i \cap W| \leq i$. We claim that every DNC function computes an infinite subset of A.

Let f be any DNC function. By a classical theorem about DNC functions (see Bienvenu et al. [1] for a proof), f computes a function $g(\cdot, \cdot, \cdot)$ such that whenever $|W_e| \leq n$, then $g(e, n, i) \in X_i \setminus W_e$. For each i, let e_i be the index of the c.e.

set $W_{e_i} = W \cap X_i$, and let $n_i = g(e_i, i, i)$. Since $|X_i \cap W| \leq i$, $|W_{e_i}| \leq i$, hence $n_i = g(e_i, i, i) \notin W_{e_i} = X_i \setminus W$. We then have two cases.

- Case 1: $n_i \in F_i$ for infinitely many i's. One can f-computably find infinitely many of them since $\mu_{\emptyset'}$ is left-c.e. and the sequence of the n's is f-computable. Therefore, one can f-computably find an infinite subset of $\bigcup_i F_i \setminus W = A$.
- Case 2: $n_i \in F_i$ for only finitely many i's. Then the sequence of the n_i's dominates the modulus function $\mu_{\emptyset'}$, and therefore computes the halting set. Since the set A is Δ^0_2, f computes an infinite subset of A.

We now detail the construction of the c.e. set W. In what follows, interpret Φ_e as a partial computable sequence of finite sets such that if $\Phi_e(x)$ and $\Phi_e(x+1)$ both halt, then $max(\Phi_e(x)) < min(\Phi_e(x+1))$. We need to satisfy the following requirements for each $e, k \in \omega$:

$$\mathscr{R}_{e,k}: \quad [\Phi_e \text{ total } \wedge (\forall i)(\forall^\infty x)(\Phi_e(x) \cap X_i = \emptyset)] \rightarrow (\exists x)[|\Phi_e(x)| > k \vee \Phi_e(x) \subseteq W]$$

We furthermore want to ensure that $|X_i \cap W| \leq i$ for each i. We can prove by induction over k that if $\mathscr{R}_{e,\ell}$ is satisfied for each $\ell \leq k$, then the set $A = \bigcup_i F_i \setminus W$ admits no computable k-enumeration. The case $k = 1$ is trivial, since if Φ_e is total and has an infinite intersection with X_i for some $i \in \omega$, then it intersects $X_i \setminus F_i$, hence intersects $\overline{A}$. For the case $k \geq 1$, if Φ_e is total, and intersects infinitely many times X_i for some $i \in \omega$, then by a finite modification, one can compute a $(k-1)$-enumeration $E_0 < E_1 < \dots$ of A by setting $E_n = \Phi_e(n) \setminus X_i$, and apply the induction hypothesis.

We now explain how to satisfy $\mathscr{R}_{e,k}$ for each $e, k \in \omega$. For each pair of indices $e, k \in \omega$, let $i_{e,k} = \sum_{\langle e',k' \rangle \leq \langle e,k \rangle} k'$. A strategy for $\mathscr{R}_{e,k}$ *requires attention* at stage $s > \langle e, k \rangle$ if $\Phi_{e,s}(x) \downarrow$, $|\Phi_{e,s}(x)| \leq k$, and $\Phi_{e,s}(x) \subseteq \bigcup_{j \geq i_{e,k}} X_j$. Then, the strategy enumerates all the elements of $\Phi_{e,s}$ in W, and is declared *satisfied*, and will never require attention again. First, notice that if Φ_e is total, outputs k-sets, and meets finitely many times each X_i, then it will require attention at some stage s and will be declared satisfied. Therefore each requirement $\mathscr{R}_{e,k}$ is satisfied. Second, suppose for the sake of contradiction that $|X_i \cap W_i| > i$ for some i. Let s be the a stage at which it happens, and let $\langle e, k \rangle < s$ be the maximal pair such that $\mathscr{R}_{e,k}$ has enumerated some element of X_i in W. In particular, $i_{e,k} \leq i$. Since the strategy for $\mathscr{R}_{e',k'}$ enumerates at most k' elements in W,

$$\sum_{\langle e',k' \rangle \leq \langle e,k \rangle} k' \geq |X_i \cap W_i| > i \geq i_{e,k} = \sum_{\langle e',k' \rangle \leq \langle e,k \rangle} k'$$

Contradiction. □

C ADS and Dependent Hyperimmunity

Proof (Proof of Lemma 30). We first show that if A_0 and A_1 are dependently X-hyperimmune then both A_0 and A_1 are X-hyperimmune. Let $F_0, F_1, \dots$ be a X-c.e. array. Let $\varphi(U, V)$ be the $\Sigma^{0,X}_1$ formula which holds if $U = F_i$ for some $i \in \omega$. The formula $\varphi(U,V)$ is essential, therefore there $\varphi(R,S)$ holds for some finite set $R \subseteq \overline{A}_0$ and $S \subseteq \overline{A}_1$. In particular, $R = F_i$ for some $i \in \omega$, therefore $F_i \subseteq \overline{A}_0$ and A_0 is hyperimmune. Similarly, the $\Sigma^{0,X}_1$ formula $\psi(U,V)$ which holds if $V = F_i$ for some $i \in \omega$ witnesses that A_1 is hyperimmune.

We now prove that if A_0 and A_1 are X-co-c.e. and X-hyperimmune, then the pair A_0,A_1 is dependently X-hyperimmune. Let $\varphi(U,V)$ be an essential $\Sigma^{0,X}_1$ formula. Define an X-c.e. sequence of sets $F_0 < F_1 < \dots$ such that for every $i \in \omega$, there is some $R < F_i$ such that $\varphi(R,F_i)$ holds and $R \subseteq \overline{A}_0$. First, notice that the sequence is X-c.e. since A_0 is X-co-c.e. Second, we claim that the sequence is infinite. To see this, define an X-c.e. array $E_0 < F_1 < \dots$ such that for every $i \in \omega$, there is some finite set $S > E_i$ such that $\psi(E_i,S)$ holds. The array is infinite since $\psi(U,V)$ is essential. Since A_0 is X-hyperimmune, there are infinitely many i's such that $E_i \subseteq \overline{A}_0$. Last, by X-hyperimmunity of A_1, there is some $i \in \omega$ such that $F_i \subseteq \overline{A}_1$. By definition of F_i, there is some $R \subseteq \overline{A}_0$ such that $\varphi(R,F_i)$ holds. □

Proof (Proof of Theorem 22). Fix an enumeration $\varphi_0(U,V), \varphi_1(U,V), \dots$ of all Σ^0_1 formulas. The construction of the function f is done by a finite injury priority argument with a movable marker procedure. We want to satisfy the following scheme of requirements for each e, where $A_i = \{x : \lim_s f(x,s) = i\}$:

$$\mathscr{R}_e : \varphi_e(U,V) \text{ essential} \rightarrow (\exists R \subseteq_{fin} A_0)(\exists S \subseteq_{fin} A_1)\varphi_e(R,S)$$

The requirements are given the usual priority ordering. We proceed by stages, maintaining two sets A_0,A_1 which represent the limit of the function f. At stage 0, $A_{0,0} = A_{1,0} = \emptyset$ and f is nowhere defined. Moreover, each requirement $\mathscr{R}_e$ is given a movable marker m_e initialized to 0.

A strategy for $\mathscr{R}_e$ *requires attention at stage* $s+1$ if $\varphi_e(R,S)$ holds for some $R < S \subseteq (m_e, s]$. The strategy sets $A_{0,s+1} = (A_{0,s} \setminus (m_e, min(S)) \cup [min(S), s]$ and $A_{1,s+1} = (A_{1,s} \setminus [min(S), s]) \cup (m_e, min(S))$. Note that $(m_e, min(S)) \cap [min(S), s] = \emptyset$ since $R < S$. Then it is declared *satisfied* and does not act until some strategy of higher priority changes its marker. Each marker $m_{e'}$ of strategies of lower priorities is assigned the value $s+1$.

At stage $s+1$, assume that $A_{0,s} \cup A_{1,s} = [0,s)$ and that f is defined for each pair over $[0,s)$. For each $x \in [0,s)$, set $f(x,s) = i$ for the unique i such that $x \in A_{i,s}$. If some strategy requires attention at stage $s+1$, take the least one and satisfy it. If no such requirement is found, set $A_{0,s+1} = A_{0,s} \cup \{s\}$ and $A_{1,s+1} = A_{1,s}$. Then go to the next stage. This ends the construction.

Each time a strategy acts, it changes the markers of strategies of lower priority, and is declared satisfied. Once a strategy is satisfied, only a strategy of higher priority can injure it. Therefore, each strategy acts finitely often and the markers stabilize. It follows that the A's also stabilize and that f is a stable function.

Claim. For every $x < y < z$, $f(x,y) = 1 \wedge f(y,z) = 1 \rightarrow f(x,z) = 1$

Proof. Suppose that $f(x,y) = 1$ and $f(y,z) = 1$ but $f(x,z) = 0$. By construction of f, $x \in A_{0,z}$, $x \in A_{1,y}$ and $y \in A_{1,z}$. Let $s \leq z$ be the last stage such that $x \in A_{1,s}$. Then at stage $s+1$, some strategy $\mathscr{R}_e$ receives attention and moves x to $A_{0,s+1}$ and therefore moves $[x,s]$ to $A_{0,s+1}$. In particular $y \in A_{0,s+1}$ since $y \in [x,s]$. Moreover, the strategies of lower priority have had their marker moved to $s+1$ and therefore will never move any element below s. Since $f(y,z) = 1$, then $y \in A_{1,z}$. In particular, some strategy $\mathscr{R}_i$ of higher priority moved y to $A_{1,t+1}$ at stage $t+1$ for some $t \in (s,z)$. Since $\mathscr{R}_i$ has a higher priority, $m_i \leq m_e$, and since y is moved to $A_{1,t+1}$, then so is $[m_i, y]$, and in particular $x \in A_{1,t+1}$ since $m_i \leq m_e \leq x \leq y$. This contradicts the maximality of s.

Claim. For every $e \in \omega$, $\mathscr{R}_e$ is satisfied.

Proof. By induction over the priority order. Let s_0 be a stage after which no strategy of higher priority will ever act. By construction, m_e will not change after stage s_0. If $\varphi_e(U,V)$ is essential, then $\varphi_e(R,S)$ holds for two sets $m_e < R < S$. Let $s = 1 + max(s_0, S)$. The strategy $\mathscr{R}_e$ will require attention at some stage before s, will receive attention, be satisfied and never be injured.

This last claim finishes the proof. ☐

Proof. If for every $x \in X$ and almost every $y \in X$, $f(x,y) \neq i$, then we can $X \oplus Z$-compute an infinite f-thin subset $Y \subseteq X$, contradicting our assumption. Let $x \in X$ be such that the $X \oplus Z$-computable set $Y = \{y \in X : f(x,y) = i\}$ is infinite. The condition $d = (F_0, \dots, F_{i-1}, F_i \cup \{x\}, F_{i+1}, \dots, F_{k-1}, Y)$ is the desired extension.

Proof (Proof of Theorem 27). It suffices to prove that for every $\Sigma^{0,Z}_1$ formula $\varphi(G,U,V)$ and every $i \in \omega$, the following class is Lebesgue null.

$$\mathscr{S} = \{X : [\varphi(X,U,V) \text{ is essential }] \wedge (\forall R, S \subseteq_{\texttt{fin}} \omega)\varphi(X,R,S) \to R \not\subseteq \overline{A}_0 \vee S \not\subseteq \overline{A}_1\}$$

Suppose it is not the case. There exists $\sigma \in 2^{<\omega}$ such that

$$\mu\{X \in \mathscr{S} : \sigma \prec X\} > 0.8 \cdot 2^{-|\sigma|}$$

Define

$$\psi(U,V) = [\mu\{X \succ \sigma : (\exists \tilde{U} \subseteq U)(\exists \tilde{V} \subseteq V)\varphi(X,\tilde{U},\tilde{V})\} > 0.6 \cdot 2^{-|\sigma|}]$$

By compactness, the formula $\psi(U,V)$ is $\Sigma^{0,Z}_1$.

Lemma 32. *$\psi(U,V)$ is essential.*

Proof. Suppose it is not. Then, there exists some $x \in \omega$, such that for every $n \in \omega$, there is some $y_n \in \omega$ such that $\psi([x,n],[y_n,+\infty))$ does not hold. Let $\mathscr{P}(X,n,y_n)$ be the formula

$$(\forall \tilde{U} \subseteq [x,n])(\forall \tilde{V} \subseteq [y_n,+\infty))\neg\varphi(X,\tilde{U},\tilde{V})$$

Unfolding the definition of $\neg\psi([x,n],[y_n,+\infty))$,

$$\mu\{X \succ \sigma : \mathscr{P}(X,n,y_n)\} > 0.4 \cdot 2^{-|\sigma|}$$

Then, by Fatou's lemma,

$$\mu\{X \succ \sigma : (\exists^\infty n)\mathscr{P}(X, n, y_n)\} > 0.2 \cdot 2^{-|\sigma|}$$

Since whenever $\mathscr{P}(X, n, y_n)$ holds, so does $\mathscr{P}(X, n-1, y_n)$,

$$\mu\{X \succ \sigma : (\forall n)(\exists y)\mathscr{P}(X, n, y)\} > 0.2 \cdot 2^{-|\sigma|}$$

Therefore

$$\mu\{X \succ \sigma : \varphi(X, U, V) \text{ is essential }\} \leq 0.8 \cdot 2^{-|\sigma|}$$

Contradicting our assumption. This finishes the lemma.

By Lemma 32 and by dependent Z-hyperimmunity of A_0, A_1, there exists some finite sets $R \subseteq \overline{A}_0$ and $S \subseteq \overline{A}_1$ such that $\psi(R, S)$ holds. For every R, S such that $\psi(R, S)$ holds, there exists some $X \in \mathscr{S}$ and some $\tilde{R} \subseteq R$ and $\tilde{S} \subseteq S$ such that $\varphi(X, \tilde{R}, \tilde{S})$ holds. By definition of $X \in \mathscr{S}$, $\tilde{R} \not\subseteq \overline{A}_0$ or $\tilde{S} \not\subseteq \overline{A}_1$ and therefore either $R \not\subseteq \overline{A}_0$ or $S \not\subseteq \overline{A}_1$. Contradiction. □