Partial orders and immunity in reverse mathematics

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Abstract. We identify computability-theoretic properties enabling us to separate various statements about partial orders in reverse mathematics. We obtain simpler proofs of existing separations, and deduce new compound ones. This work is part of a larger program of unification of the separation proofs of various Ramsey-type theorems in reverse mathematics in order to obtain a better understanding of the combinatorics of Ramsey's theorem and its consequences. We also answer a question of Murakami, Yamazaki and Yokoyama about pseudo Ramsey's theorem for pairs.

1 Introduction

Many theorems of "ordinary" mathematics are of the form

$$(\forall X)[\Phi(X) \rightarrow (\exists Y)\Psi(X, Y)]$$

where Φ and Ψ are arithmetic formulas. They can be seen as *mathematical problems*, whose *instances* are sets X such that $\Phi(X)$ holds, and whose *solutions* to X are sets Y such that $\Psi(X, Y)$ holds. For example, König's lemma asserts that every infinite, finitely branching tree admits an infinite path through it.

There exist many ways to calibrate the strength of a mathematical problem. Among them, *reverse mathematics* is a vast foundational program that seeks to determine the weakest axioms necessary to prove ordinary theorems. It uses the framework of subsystems of second-order arithmetic, within the base theory RCA_0, which can be thought of as capturing *computable mathematics*. An ω-*structure* is a structure whose first-order part consists of the standard integers. The ω-models of RCA_0 are those whose second-order part is a *Turing ideal*, that is, a collection of sets $\mathscr{S}$ downward-closed under the Turing reduction and closed under the effective join.

In this setting, an ω-model $\mathscr{M}$ satisfies a mathematical problem P if every P-instance in $\mathscr{M}$ has a solution in $\mathscr{M}$. A standard way of proving that a problem P does not imply another problem Q consists of creating an ω-model $\mathscr{M}$ satisfying P but not Q. Such a model is usually constructed by taking a ground Turing ideal, and extending it by iteratively adding solutions to its P-instances. However, while taking the closure of the collection $\mathscr{M} \cup \{Y\}$ to obtain a Turing ideal, one may add solutions to Q-instances as well. The whole difficulty of this construction consists of finding the right computability-theoretic notion preserved by P but not by Q.

We conduct a program of identification of the computability-theoretic properties enabling us to distinguish various Ramsey-type theorems in reverse mathematics,

but also under computable and Weihrauch reducibilities. This program puts emphasis on the interplay between computability theory and reverse mathematics, the former providing tools to separate theorems in reverse mathematics over standard models, and the latter exhibiting new computability-theoretic properties.

Among the theorems studied in reverse mathematics, the ones coming from Ramsey's theory play a central role. Their strength are notoriously hard to gauge, and required the development of involved computability-theoretic frameworks. Perhaps the most well-known example is Ramsey's theorem.

Definition 1 (Ramsey's theorem). *A subset H of ω is* homogeneous *for a coloring $f : [\omega]^n \to k$ (or f-homogeneous) if each n-tuple from H is given the same color by f. RT^n_k is the statement "Every coloring $f : [\omega]^n \to k$ has an infinite f-homogeneous set".*

Jockusch [13] conducted a computational analysis of Ramsey's theorem. He proved in particular that RT^n_k implied the existence of the halting set whenever $n \geq 3$. There has been a lot of literature around the strength of Ramsey's theorem for pairs [4,7,11,22] and its consequences [3,5,12]. We focus on some mathematical statements about partial orders which are consequences of Ramsey's theorem for pairs.

Definition 2 (Chain-antichain). *A* chain *in a partial order $(P, \leq_P)$ is a set $S \subseteq P$ such that $(\forall x, y \in S)(x \leq_P y \lor y \leq_P x)$. An* antichain *in P is a set $S \subseteq P$ such that $(\forall x, y \in S)(x \neq y \to x|_P y)$ (where $x|_P y$ means that $x \not\leq_P y \land y \not\leq_P x$). CAC is the statement "every infinite partial order has an infinite chain or an infinite antichain."*

The chain-antichain principle was introduced by Hirschfeldt and Shore [12] together with the ascending descending sequence (ADS). They studied extensively cohesive and stable versions of the statements, and proved that CAC is computationally weak, in that it does not even imply the existence of a diagonally noncomputable function. However, their proof has an ad-hoc flavor, in that it is a direct separation involving the two statements. Later, Lerman, Solomon and Towsner [17] separated ADS from CAC over ω-models by using an involved iterated forcing argument.

In this paper, we revisit the two proofs and emphasize the combinatorial nature of the principles by identifying the computability-theoretic properties separating them. Those properties happen to be very natural and coincide on co-c.e. sets with some well-known computability-theoretic notions, namely, immunity and hyperimmunity. The proof of the separation of ADS from CAC is significantly simpler and more modular, as advocated by the author in [20]. Last, we give a simpler separation of two versions of stability for the chain-antichain principles over computable reducibility, which was previously proven by Astor et al. [1] by the means of a mutually dependent elaborate notion of forcing.[1]

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[1] This paper is an extended version of a conference paper of the same name published in CiE 2016.

1.1 Notation and definitions

Given two sets A and B, we denote by $A < B$ the formula $(\forall x \in A)(\forall y \in B)[x < y]$ and by $A \subseteq^* B$ the formula $(\forall^\infty x \in A)[x \in B]$, meaning that A is included in B *up to finitely many elements*. A *Mathias condition* is a pair (F, X) where F is a finite set, X is an infinite set and $F < X$. A condition (F_1, X_1) *extends* (F, X) (written $(F_1, X_1) \leq (F, X)$) if $F \subseteq F_1$, $X_1 \subseteq X$ and $F_1 \smallsetminus F \subset X$. A set G *satisfies* a Mathias condition (F, X) if $F \subset G$ and $G \smallsetminus F \subseteq X$.

2 Preservation of properties for co-c.e. sets

Ramsey's theorem for k colors has a deeply disjunctive nature. One cannot know in a finite amount of time whether a coloring will admit an infinite homogeneous set for a fixed color, and one must therefore build multiple homogeneous sets simultaneously, namely, one for each color. This disjunction was exploited by the author to show for example that ADS does not preserve 2 hyperimmunities simultaneously, whereas the Erdős-Moser theorem does [20]. This idea was also used in the context of computable reducibility to show that RT^2_{k+1} does not computably reduce to RT^2_k whenever $k \geq 1$, by showing that RT^2_k preserves 2 among $k + 1$ hyperimmunities simultaneously whereas RT^2_{k+1} does not [21]. In this section, we shall see that this disjunctive flavor disappears whenever considering co-c.e. sets. In particular, RT^2_2 admits preservation of countably many hyperimmune co-c.e. sets simultaneously.

Definition 3 (Hyperimmunity). *An* array *is a sequence of mutually disjoint finitely coded sets. An array* $F_0, F_1, \ldots$ traces *a set* A *if* $F_i \cap A \neq \emptyset$ *for every* $i \in \omega$. *A set* A *is* X-hyperimmune *if it is not traced by any* X-computable array.

Equivalently, a set is X-hyperimmune if its principal function is not dominated by any X-computable function, where the *principal function* p_A of a set $A = \{x_0 < x_1 < \ldots\}$ is defined by $p_A(i) = x_i$.

Definition 4 (Preservation of hyperimmunity for co-c.e. sets). *A* Π^1_2 *statement* P admits *preservation of hyperimmunity for co-c.e. sets if for every set* Z, *every sequence of* Z-co-c.e. Z-hyperimmune *sets* $A_0, A_1, \ldots$ *and every* P-*instance* $X \leq_T Z$, *there is a solution* Y *to* X *such that the* A's *are* $Y \oplus Z$-*hyperimmune*.

The remainder of this section is devoted to the proof that preservation of 1 hyperimmunity and preservation of (countable) hyperimmunity for co-c.e. sets coincide.

Lemma 5. *If* A_0 *and* A_1 *are co-c.e. hyperimmune sets, then so is* $A_0 \cup A_1$.

Proof. If A_0 and A_1 are co-c.e., then so is $A_0 \cup A_1$. We now prove that $A_0 \cup A_1$ is hyperimmune. Let $F_0, F_1, \ldots$ be an array tracing $A_0 \cup A_1$. If for infinitely many s, $F_s \cap A_0 = \emptyset$, then we can $\vec{F}$-computably find infinitely many such s since A_0 is co-c.e. Note that for any such s, $F_s \cap A_1 \neq \emptyset$. Since A_1 is hyperimmune, it follows that $\vec{F}$ is not computable. If for all but finitely many s, $F_s \cap A_0 \neq \emptyset$, then we can $\vec{F}$-compute an array tracing A_0, and therefore $\vec{F}$ is not computable by hyperimmunity of A_0.

Lemma 6. *Let $A_0, A_1, \ldots$ be a (non-effective) listing of co-c.e. hyperimmune sets. There is a hyperimmune set B such that every array tracing any A_i computes an array tracing B.*

Proof. Let $B = \{\langle x, y \rangle : x \in A_0 \wedge y \in \bigcup_{j \leq x} A_j\}$. For every array $F_0, F_1, \ldots$ tracing A_n, and every $x \in A_0$ such that $x \geq n$, the $\vec{F}$-computable array $G_0, G_1, \ldots$ defined by $G_i = \{\langle x, y \rangle : y \in F_i\}$ traces B. We now prove that B is hyperimmune. Let $F_0, F_1, \ldots$ be an array tracing B. We need to prove that $\vec{F}$ is not computable. Let $G_0, G_1, \ldots$ be the $\vec{F}$-computable sequence defined inductively as follows: $G_0 = \{x_0\}$ for some $x_0 \in A_0$. Assume we have defined G_i, and let $x_i = \max G_i$. We search for stages $t > s > i$ such that

$$(\forall \langle x, y \rangle \in F_s)[x \leq x_i \to (x \notin A_{0,t} \vee y \notin \bigcup_{j \leq x} A_{j,t})]$$

where $A_{j,s}$ is the approximation of the co-c.e. set A_j at stage s. If we find such a stage, we let $G_{i+1} = \{x > x_i : \langle x, y \rangle \in F_s\}$. We have two cases. In the first case, the sequence of the G's is infinite. By construction, G is an $\vec{F}$-computable array tracing A_0. It follows that $\vec{F}$ is not computable by hyperimmunity of A_0. In the second case, the sequence of the G's is finite. Let G_i be its last element and $x_i = \max G_i$. Then, for every $t > s \geq i$, there is some $\langle x, y \rangle \in F_s$, such that $x \leq x_i$ and $y \in \bigcup_{j \leq x} A_j$. For every $s \geq i$, let $H_s = \{y : (\exists \langle x, y \rangle \in F_s)[x \leq x_i]\}$. The $\vec{F}$-computable array $\vec{H}$ traces $\bigcup_{j \leq x_i} A_j$. However, each A_j is co-c.e. and hyperimmune, so by Lemma 5, $\bigcup_{j \leq x_i} A_j$ is hyperimmune. It follows that $\vec{F}$ is not computable.

Lemma 7. *If P admits preservation of 1 hyperimmunity, then it admits preservation of hyperimmunity for co-c.e. sets.*

Proof. Fix a set Z, a countable sequence of Z-co-c.e. Z-hyperimmune sets $A_0, A_1, \ldots,$ and a Z-computable P-instance X. By a relativization of Lemma 6, there is a Z-hyperimmune set B such that every array tracing any A_i Z-computes an array tracing B. By preservation of 1 hyperimmunity of P, there is a solution Y to X such that B is $Y \oplus Z$-hyperimmune. It follows that each A_i is $Y \oplus Z$-hyperimmune.

Corollary 8. RT^2_2 *admits preservation of hyperimmunity for co-c.e. sets.*

Proof. The author proved in [20] that RT^2_2 admits preservation of 1 hyperimmunity. The corollary now follows from Lemma 7.

3 CAC **and constant-bound immunity**

Hirschfeldt and Shore [12] separated CAC from DNC in reverse mathematics by a direct construction. DNC is the statement asserting, for every set X, the existence of a function f such that $f(e) \neq \Phi^X_e(e)$ for every e. In this section, we extract the core of the combinatorics of their forcing argument to exhibit a computability-theoretic property separating the two notions, namely, constant-bound immunity.

Definition 9 (Constant-bound immunity). *A k-enumeration (k-enum) of a set A is an infinite sequence of k-sets $F_0 < F_1 < \ldots$ such that for every $i \in \omega$, $F_i \cap A \neq \emptyset$. A constant-bound enumeration (c.b-enum) of a set A is a k-enumeration of A for some $k \in \omega$. A set A is k-immune (c.b-immune) relative to X if it admits no X-computable k-enumeration (c.b-enumeration).*

In particular, 1-immunity coincides with the standard notion of immunity. Also note that one can easily create a c.b-immune set computing no effectively immune set. The following lemma shows that c.b-immunity and immunity coincide for co-c.e. sets.

Lemma 10. *An X-co-c.e. set A is c.b-immune relative to X iff it is X-immune.*

Proof. We first prove that if A is not X-immune, then it is not c.b-immune relative to X. Let $W = \{w_0 < w_1 < \ldots\}$ be an infinite X-computable infinite subset of A. The sequence $F_0, F_1, \ldots$ defined by $F_i = \{w_i\}$ is an X-computable 1-enum (hence c.b-enum) of A. Therefore, A is not c.b-immune relative to X.

We now show by induction over k if A is X-co-c.e. and has an X-computable k-enumeration $F_0, F_1, \ldots$ then it has an infinite X-computable subset. If $k = 1$, then it is already an infinite subset of A. Suppose now that $k \geq 2$. If there are infinitely many $i \in \omega$ such that $min(F_i) \in \bar{A} \neq \emptyset$, then since A is X-co-c.e., one can find an X-computable infinite set S of such i's. The sequence $\{F_i \smallsetminus min(F_i) : i \in S\}$ is an X-computable $(k-1)$-enumeration of A, and by induction hypothesis, there is an X-computable subset of A. If there are only finitely many such i's, then the sequence $\{min(F_i) : i \in \omega)\}$ is, up to finite changes, an infinite X-computable subset of X. $\qquad\square$

Definition 11 (Preservation of c.b-immunity). *A Π_2^1 statement P admits preservation of c.b-immunity if for every set Z, every set A which is c.b-immune relative to Z, and every P-instance $X \leq_T Z$, there is a solution Y to X such that A is c.b-immune relative to $Y \oplus Z$.*

We can easily relate the notion of preservation of c.b-immunity with the existing notion of constant-bound enumeration avoidance defined by Liu [18] to separate RT_2^2 from WWKL over RCA_0.

Definition 12 (Constant-bound enumeration avoidance). *A k-enumeration (k-enum) of a class $\mathscr{C} \subseteq 2^\omega$ is an infinite sequence of k-sets $F_0 < F_1 < \ldots$ of strings such that for every $i \in \omega$, every string $\sigma \in F_i$ is of length i, and $\mathscr{C} \cap [F_i] \neq \emptyset$, where $[F_i]$ is the clopen set of all sequences extending some string in F_i. A constant-bound enumeration (c.b-enum) of $\mathscr{C}$ is a k-enumeration of $\mathscr{C}$ for some $k \in \omega$.*

A Π_2^1 statement P admits constant-bound enumeration avoidance if for every set Z, every class $\mathscr{C} \subseteq 2^\omega$ with no Z-computable c.b-enum, and every P-instance $X \leq_T Z$, there is a solution Y to X such that $\mathscr{C}$ has no $Y \oplus Z$-computable c.b-enum.

Lemma 13. *If P admits preservation of c.b-immunity, then it admits constant-bound enumeration avoidance.*

Proof. Fix a non-empty class $\mathscr{C} \subseteq 2^\omega$, and let $A = \{\sigma : \mathscr{C} \cap [\sigma] \neq \emptyset\}$. We claim that the degrees of the c.b-enums of A and of $\mathscr{C}$ coincide. Any c.b-enum of $\mathscr{C}$ is a c.b-enum of A. Conversely, let $F_0 < F_1 < \ldots$ be a c.b-enum of $\mathscr{C}$. We can computably thin it out and normalize it into an enumeration $E_0 < E_1 < \ldots$ such that $|\sigma| = i$ for every $\sigma \in E_i$. $\square$

Hirschfeldt and Shore [12] proved that CAC is equivalent to the existence of homogeneous sets for semi-transitive colorings. A coloring $f : [\mathbb{N}]^2 \to 2$ is *semi-transitive* if whenever $f(x, y) = 1$ and $f(y, z) = 1$, then $f(x, z) = 1$ for $x < y < z$. We shall use this equivalence to prove the following theorem.

Theorem 14. CAC *admits preservation of c.b-immunity.*

Proof. Let A be a set c.b-immune relative to some set Z, and let $f : [\omega]^2 \to 2$ be a Z-computable semi-transitive coloring. Assume that there is no infinite f-homogeneous set H such that A is c.b-immune relative to $H \oplus Z$, otherwise we are done. We will build two infinite sets G_0 and G_1, such that G_i is f-homogeneous for color i for each $i < 2$, and such that A is c.b-immune relative to $G_i \oplus Z$ for some $i < 2$.

The construction is done by a variant of Mathias forcing (F_0, F_1, X), where F_0 and F_1 are finite sets, and X is infinite Z-computable set such that $max(F_0, F_1) < min(X)$. Moreover, we require that for every $i < 2$ and every $x \in X$, $F_i \cup \{x\}$ is f-homogeneous for color i. A condition (E_0, E_1, Y) *extends* (F_0, F_1, X) if (E_i, Y) Mathias extends (F_i, X) for each $i < 2$. A pair of sets G_0, G_1 *satisfies* a condition $c = (F_0, F_1, X)$ if G_i is f-homogeneous for color i and satisfies the Mathias condition (F_i, X) for each $i < 2$.

Lemma 15. *For every condition $c = (F_0, F_1, X)$ and every $i < 2$, there is an extension (E_0, E_1, Y) of c such that $|E_i| > |F_i|$.*

Proof. Take any $x \in X$ such that the set $Y = \{y \in X : f(x, y) = i\}$ is infinite. Such an x must exist, otherwise the set X is limit-homogeneous for color $1 - i$ and one can Z-compute an infinite f-homogeneous set, contradicting our hypothesis. Let $E_i = F_i \cup \{x\}$ and $E_{1-i} = F_{1-i}$, and take (E_0, E_1, Y) as the desired extension.

In what follows, we interpret $\Phi_0, \Phi_1, \ldots$ as Turing functionals outputting non-empty finite sets such that if $\Phi_e^X(x)$ and $\Phi_e^X(x+1)$ both halt, $max(\Phi_e^X(x)) < min(\Phi_e^X(x+1))$. We want to satisfy the following requirements for each $e_0, k_0, e_1, k_1 \in \omega$:

$$\mathscr{R}_{e_0, k_0, e_1, k_1} : \quad \mathscr{R}_{e_0, k_0}^{G_0} \quad \vee \quad \mathscr{R}_{e_1, k_1}^{G_1}$$

where $\mathscr{R}_{e,k}^G$ is the requirement

$$(\exists x)\left(\Phi_e^{G \oplus Z}(x) \uparrow \vee |\Phi_e^{G \oplus Z}(x)| > k \vee \Phi_e^{G \oplus Z}(x) \cap A = \emptyset\right).$$

In other words, $\mathscr{R}_{e,k}^G$ asserts that $\Phi_e^{G \oplus Z}$ is not a k-enumeration of A. A condition c *forces* a formula $\varphi(G_0, G_1)$ if $\varphi(G_0, G_1)$ holds for every pair of *infinite* sets G_0, G_1 satisfying c.

Lemma 16. *For every condition c and every vector of indices $e_0, k_0, e_1, k_1 \in \omega$, there is an extension d of c forcing $\mathscr{R}_{e_0, k_0, e_1, k_1}$.*

Proof. Fix a condition $c = (F_0, F_1, X)$, and let $P_0, P_1, \ldots$ be an Z-computable sequence of sets where $P_n = \Phi_{e_0}^{(F_0 \cup E_0) \oplus Z}(x_0) \cup \Phi_{e_1}^{(F_1 \cup E_1) \oplus Z}(x_1)$ for a pair of sets $E_1 < E_0 \subseteq X$ and some $x_0, x_1 \in \omega$ such that E_0 is f-homogeneous for color 0, $E_1 \cup \{y\}$ is f-homogeneous for color 1 for each $y \in E_0$, and for each $i < 2$, $\max(P_{n-1}) < \min(\Phi_{e_i}^{(F_i \cup E_i) \oplus Z}(x_i))$ and $|\Phi_{e_i}^{(F_i \cup E_i) \oplus Z}(x_i)| \leq k_i$. We have two cases.

- Case 1: the sequence of the P's is finite and is defined, say to level $n-1$. If there is a pair of infinite sets G_0, G_1 satisfying c and some $x_1 \in \omega$ such that $\Phi_{e_1}^{G_1 \oplus Z}(x_1) \downarrow$, $\max(P_{n-1}) < \min(\Phi_{e_1}^{G_1 \oplus Z}(x_1))$, and $|\Phi_{e_1}^{G_1 \oplus Z}(x_1)| \leq k_1$, then let $E_1 \subseteq G_1$ be such that $F_1 \cup E_1$ is an initial segment of G_1 for which $\Phi_{e_1}^{(F_1 \cup E_1) \oplus Z}(x_1) \downarrow$. The set $Y = \{y \in X : E_1 \cup \{y\}$ is f-homogeneous for color 1 $\}$ is a superset of G_1, hence is infinite. The condition $d = (F_0, F_1 \cup E_1, Y)$ is an extension of c forcing $\mathscr{R}_{e_0, k_0}^{G_0}$, hence forcing $\mathscr{R}_{e_0, k_0, e_1, k_1}$. If there is no such pair of infinite sets G_0, G_1, then the condition c already forces $\mathscr{R}_{e_1, k_1}^{G_1}$, hence $\mathscr{R}_{e_0, k_0, e_1, k_1}$.
- Case 2: the sequence of the P's is infinite. By c.b-immunity of A relative to Z, $P_n \cap A = \emptyset$ for some $n \in \omega$. Let $E_1 < E_0 \subseteq X$ and $x_0, x_1 \in \omega$ witness the existence of P_n. If $Y_0 = \{y \in X : E_0 \cup \{y\}$ is f-homogeneous for color 0 $\}$ is infinite, then the condition $(F_0 \cup E_0, F_1, Y_0)$ is an extension of c forcing $\mathscr{R}_{e_0, k_0}^{G_0}$. If Y_0 is finite, then for almost every $y \in X$, there is some $x_y \in E_0$ such that $f(x_y, y) = 1$, and by transitivity of f for color 1, $E_1 \cup \{y\}$ is f-homogeneous for color 1. Indeed, E_1 is f-homogeneous for color 1 and for each $x \in E_1$, $f(x, x_y) = f(x_y, y) = 1$. In this case, $(F_0, F_1 \cup E_1, Y_1)$ is an extension of c forcing $\mathscr{R}_{e_1, k_1}^{G_1}$, for some $Y_1 =^* X$. In both cases, there is an extension of c forcing $\mathscr{R}_{e_0, k_0, e_1, k_1}$.

Let $\mathscr{F} = \{c_0, c_1, \ldots\}$ be a sufficiently generic filter containing $(\emptyset, \emptyset, \omega)$, where $c_s = (F_{0,s}, F_{1,s}, X_s)$. The filter $\mathscr{F}$ yields a pair of sets G_0, G_1 defined by $G_i = \bigcup_s F_{i,s}$. By Lemma 15, the sets G_0 and G_1 are both infinite, and by Lemma 16, the set A is c.b-immune relative to $G_i \oplus Z$ for some $i < 2$. This completes the proof of Theorem 14. □

Theorem 17. DNC *does not admit preservation of c.b-immunity.*

Proof. Let $\mu_{\emptyset'}$ be the modulus function of $\emptyset'$, that is, such that $\mu_{\emptyset'}(x)$ is the minimum stage s at which $\emptyset'_s \restriction x = \emptyset' \restriction x$. The sketch of the proof is the following:

Computably split ω into countably many columns $X_0, X_1, \ldots$ of infinite size. For example, set $X_i = \{\langle i, n \rangle : n \in \omega\}$ where $\langle \cdot, \cdot \rangle$ is a bijective function from ω^2 to ω. For each i, let F_i be the set of the $\mu_{\emptyset'}(i)$ first elements of X_i. The sequence $F_0, F_1, \ldots$ is $\emptyset'$-computable. Assume for now that we have defined a c.e. set W such that the Δ_2^0 set $A = \bigcup_i F_i \smallsetminus W$ is c.b-immune, and such that $|X_i \cap W| \leq i$. We claim that every DNC function computes an infinite subset of A.

Let f be any DNC function. By a classical theorem about DNC functions (see Bienvenu et al. [2] for a proof), f computes a function $g(\cdot, \cdot, \cdot)$ such that whenever $|W_e| \leq n$, then $g(e, n, i) \in X_i \smallsetminus W_e$. For each i, let e_i be the index of the c.e. set $W_{e_i} = W \cap X_i$, and let $n_i = g(e_i, i, i)$. Since $|X_i \cap W| \leq i$, $|W_{e_i}| \leq i$, hence $n_i = g(e_i, i, i) \in X_i \smallsetminus W_{e_i}$, which implies $n_i \in X_i \smallsetminus W$. We then have two cases.

– Case 1: $n_i \in F_i$ for infinitely many i's. One can f-computably find infinitely many of them since $\mu_{\emptyset'}$ is left-c.e. and the sequence of the n's is f-computable. Therefore, one can f-computably find an infinite subset of $\bigcup_i F_i \smallsetminus W = A$.

– Case 2: $n_i \in F_i$ for only finitely many i's. Then the sequence of the n_i's dominates the modulus function $\mu_{\emptyset'}$, and therefore computes the halting set. Since the set A is Δ^0_2, f computes an infinite subset of A.

We now detail the construction of the c.e. set W. In what follows, interpret Φ_e as a partial computable sequence of finite sets such that if $\Phi_e(x)$ and $\Phi_e(x+1)$ both halt, then $\max(\Phi_e(x)) < \min(\Phi_e(x+1))$. We need to satisfy the following requirements for each $e, k \in \omega$:

$$\mathscr{R}_{e,k}: \quad [\Phi_e \text{ total } \wedge (\forall i)(\forall^\infty x)(\Phi_e(x) \cap X_i = \emptyset)] \to (\exists x)[|\Phi_e(x)| > k \vee \Phi_e(x) \subseteq W]$$

We furthermore want to ensure that $|X_i \cap W| \leq i$ for each i. We can prove by induction over k that if $\mathscr{R}_{e,\ell}$ is satisfied for each $\ell \leq k$, then the set $A = \bigcup_i F_i \smallsetminus W$ admits no computable k-enumeration. The case $k = 1$ is trivial, since if Φ_e is total and has an infinite intersection with X_i for some $i \in \omega$, then it intersects $X_i \smallsetminus F_i$, hence intersects $\overline{A}$. For the case $k \geq 2$, assume that Φ_e is total, and has infinite intersection with X_i for some $i \in \omega$. By our assumption that $\max(\Phi_e(x)) < \min(\Phi_e(x+1))$, for large enough n, $F_i < \Phi_e(n) \subseteq X_i$, and hence $\Phi_e(n) \subseteq \overline{A}$. Otherwise, one can compute a $(k-1)$-enumeration $E_0 < E_1 < \ldots$ of A by setting $E_n = \Phi_e(n) \smallsetminus X_i$, and apply the induction hypothesis.

We now explain how to satisfy $\mathscr{R}_{e,k}$ for each $e, k \in \omega$. For each pair of indices $e, k \in \omega$, let $i_{e,k} = \sum_{\langle e',k'\rangle \leq \langle e,k\rangle} k'$. A strategy for $\mathscr{R}_{e,k}$ *requires attention* at stage $s > \langle e, k \rangle$ if there is an $x < s$ such that $\Phi_{e,s}(x) \downarrow$, $|\Phi_{e,s}(x)| \leq k$, and $\Phi_{e,s}(x) \subseteq \bigcup_{j \geq i_{e,k}} X_j$. Then, the strategy enumerates all the elements of $\Phi_{e,s}$ in W, and is declared *satisfied*, and will never require attention again. First, notice that if Φ_e is total, outputs k-sets, and meets finitely many times each X_i, then it will require attention at some stage s and will be declared satisfied. Therefore each requirement $\mathscr{R}_{e,k}$ is satisfied. Second, suppose for the sake of contradiction that $|X_i \cap W| > i$ for some i. Let s be the stage at which it happens, and let $\langle e, k \rangle < s$ be the maximal pair such that $\mathscr{R}_{e,k}$ has enumerated some element of X_i in W. In particular, $i_{e,k} \leq i$. Since the strategy for $\mathscr{R}_{e',k'}$ enumerates at most k' elements in W,

$$\sum_{\langle e',k'\rangle \leq \langle e,k\rangle} k' \geq |X_i \cap W| > i \geq i_{e,k} = \sum_{\langle e',k'\rangle \leq \langle e,k\rangle} k'$$

Contradiction.

Corollary 18 (Hirschfeldt and Shore [12]). $\mathsf{RCA}_0 \wedge \mathsf{CAC} \nvdash \mathsf{DNC}$.

4 ADS **and pseudo Ramsey's theorem for pairs**

In this section, we answer a question of Murakami, Yamazaki and Yokoyama in [19] by proving the equivalence between the ascending descending sequence, introduced by Hirschfeldt and Shore [12] and pseudo Ramsey's theorem for pairs and two colors. This equivalence was independently obtained by Steila in [23].

Definition 19 (Ascending descending sequence). *Given a linear order* $(L, <_L)$*, an* ascending *(descending) sequence is a set S such that for every* $x <_{\mathbb{N}} y \in S$*,* $x <_L y$ $(x >_L y)$*.* ADS *is the statement "Every infinite linear order admits an infinite ascending or descending sequence".*

Pseudo Ramsey's theorem for pairs was first introduced by Friedman [9] and later studied by Friedman and Pelupessy [10], and Murakami, Yamazaki and Yokoyama in [19] who proved that it is between the chain antichain principle and the ascending descending sequence principle over RCA_0.

Definition 20 (Pseudo Ramsey's theorem). *A set* $H = \{x_0 < x_1 < \ldots\}$ *is* pseudo-homogeneous *for a coloring* $f : [\mathbb{N}]^n \to k$ *if* $f(x_i, \ldots, x_{i+n-1}) = f(x_j, \ldots, x_{j+n-1})$ *for every* $i, j \in \mathbb{N}$*.* psRT^n_k *is the statement "Every coloring* $f : [\mathbb{N}]^n \to k$ *has an infinite pseudo-homogeneous set".*

We now prove the equivalence between the two statements. Note however that the exact strength of psRT^2_k remains open in reverse mathematics whenever $k > 2$.

Theorem 21. $\mathsf{RCA}_0 \vdash \mathsf{psRT}^2_2 \leftrightarrow \mathsf{ADS}$

Proof. The direction $\mathsf{psRT}^2_2 \to \mathsf{ADS}$ is Theorem 24 in [19]. We prove that $\mathsf{ADS} \to \mathsf{psRT}^2_2$. Let $f : [\mathbb{N}]^2 \to 2$ be a coloring. The reduction is in two steps. We first define a $\Delta^{0,f}_1$ semi-transitive coloring $g : [\mathbb{N}]^2 \to 2$ such that every infinite set pseudo-homogeneous for g computes an infinite set pseudo-homogeneous for f. Then, we define a $\Delta^{0,g}_1$ linear order $h : [\mathbb{N}]^2 \to 2$ such that every infinite set pseudo-homogeneous for h computes an infinite set pseudo-homogeneous for g. We conclude by applying ADS over h.

Step 1: Define the coloring $g : [\mathbb{N}]^2 \to 2$ for every $x < y$ by $g(x, y) = 1$ if there exists a sequence $x = x_0 < \ldots < x_l = y$ such that $f(x_i, x_{i+1}) = 1$ for every $i < l$, and $g(x, y) = 0$ otherwise. The function g is a semi-transitive coloring. Indeed, suppose that $g(x, y) = 1$ and $g(y, z) = 1$, witnessed respectively by the sequences $x = x_0 < \ldots < x_m = y$ and $y = y_0 < \ldots < y_n = z$. The sequence $x = x_0 < \ldots < x_m = y_0 < \ldots < y_n = z$ witnesses $g(x, z) = 1$. We claim that every infinite set $H = \{x_0 < x_1 < \ldots\}$ pseudo-homogeneous for g computes an infinite set pseudo-homogeneous for f. If H is pseudo-homogeneous with color 0, then $f(x_i, x_{i+1}) = 0$ for each i, otherwise the sequence $x_i < x_{i+1}$ would witness $g(x_i, x_{i+1}) = 1$. Thus H is pseudo-homogeneous for f with color 0. If H is pseudo-homogeneous with color 1, then define the set $H_1 \supseteq H$ to be the set of integers in the sequences witnessing $g(x_i, x_{i+1}) = 1$ for each i. The set H_1 is $\Delta^{0,f \oplus H}_1$ and pseudo-homogeneous for f with color 1.

Step 2: Define the coloring $h : [\mathbb{N}]^2 \to 2$ for every $x < y$ by $h(x, y) = 0$ if there exists a sequence $x = x_0 < \ldots < x_l = y$ such that $g(x_i, x_{i+1}) = 0$ for every $i < l$, and $h(x, y) = 1$ otherwise. For the same reasons as for g, $h(x, z) = 0$ whenever $h(x, y) = 0$ and $h(y, z) = 0$ for $x < y < z$. We need to prove that if $h(x, z) = 0$ then either $h(x, y) = 0$ or $h(y, z) = 0$ for $x < y < z$. Let $x = x_0 < \ldots < x_l = z$ be a sequence witnessing $h(x, z) = 0$. If $y = x_i$ for some $i < l$ then the sequence $x = x_0 < \ldots < x_i = y$ witnesses $h(x, y) = 0$. If $y \neq x_i$ for every $i < l$, then there exists

some $i < l$ such that $x_i < y < x_{i+1}$. By semi-transitivity of g, either $g(x_i, y) = 0$ or $g(y, x_{i+1}) = 0$. In this case either $x = x_0 < \ldots < x_i < y$ witnesses $h(x, y) = 0$ or $y < x_{i+1} < \ldots < x_l = z$ witnesses $h(y, z) = 0$. Therefore h is a linear order. For the same reasons as for g, every infinite set pseudo-homogeneous for h computes an infinite set pseudo-homogeneous for g. This last step finishes the proof. $\qquad \square$

5   ADS **and dependent hyperimmunity**

Lerman, Solomon and Towsner [17] separated the ascending descending sequence principle from a stable version of CAC by using a very involved iterated forcing argument. According to our previous simplification of their general framework [20], we reformulate their proof in terms of preservation of dependent hyperimmunity, and extend it to pseudo Ramsey's theorem for pairs.

Definition 22 (Dependent hyperimmunity). *A formula $\varphi(U, V)$ is* essential *if for every $x \in \omega$, there is a finite set $R > x$ such that for every $y \in \omega$, there is a finite set $S > y$ such that $\varphi(R, S)$ holds. A pair of sets $A_0, A_1 \subseteq \omega$ is* dependently X-hyperimmune *if for every essential $\Sigma_1^{0,X}$ formula $\varphi(U, V)$, $\varphi(R, S)$ holds for some $R \subseteq \overline{A}_0$ and $S \subseteq \overline{A}_1$.*

In particular, if the pair A_0, A_1 is dependently hyperimmune, then A_0 and A_1 are both hyperimmune.

Definition 23 (Preservation of dependent hyperimmunity). *A Π_2^1 statement* P *admits* preservation of dependent hyperimmunity *if for every set Z, every pair of dependently Z-hyperimmune sets $A_0, A_1 \subseteq \omega$ and every* P*-instance $X \leq_T Z$, there is a solution Y to X such that A_0, A_1 are dependently $Y \oplus Z$-hyperimmune.*

A partial order $(P, \leq_P)$ is *stable* if either $(\forall i \in P)(\exists s)[(\forall j > s)(j \in P \rightarrow i \leq_P j) \vee (\forall j > s)(j \in P \rightarrow i \mid_P j)]$ or $(\forall i \in P)(\exists s)[(\forall j > s)(j \in P \rightarrow i \geq_P j) \vee (\forall j > s)(j \in P \rightarrow i \mid_P j)]$. SCAC is the restriction of CAC to stable partial orders. A simple finite injury priority argument shows that SCAC does not admit preservation of dependent hyperimmunity.

Theorem 24. *There exists a computable, stable semi-transitive coloring $f : [\omega]^2 \rightarrow 2$ such that the pair $\overline{A}_0, \overline{A}_1$ is dep. hyperimmune, where $A_i = \{x : \lim_s f(x, s) = i\}$.*

Proof. Fix an enumeration $\varphi_0(U, V), \varphi_1(U, V), \ldots$ of all Σ_1^0 formulas. The construction of the function f is done by a finite injury priority argument with a movable marker procedure. We want to satisfy the following scheme of requirements for each e, where $A_i = \{x : \lim_s f(x, s) = i\}$:

$$\mathscr{R}_e : \varphi_e(U, V) \text{ essential} \rightarrow (\exists R \subseteq_{fin} A_0)(\exists S \subseteq_{fin} A_1)\varphi_e(R, S)$$

The requirements are given the usual priority ordering. We proceed by stages, maintaining two sets A_0, A_1 which represent the limit of the function f. At stage 0, $A_{0,0} = A_{1,0} = \emptyset$ and f is nowhere defined. Moreover, each requirement $\mathscr{R}_e$ is given a movable marker m_e initialized to 0.

A strategy for $\mathscr{R}_e$ *requires attention at stage $s+1$* if $\varphi_e(R,S)$ holds for some $R < S \subseteq (m_e, s]$. The strategy sets $A_{0,s+1} = (A_{0,s} \smallsetminus (m_e, min(S)) \cup [min(S), s]$ and $A_{1,s+1} = (A_{1,s} \smallsetminus [min(S), s]) \cup (m_e, min(S))$. Note that $R \subseteq (m_e, min(S))$ since $R < S$. Then it is declared *satisfied* and does not act until some strategy of higher priority changes its marker. Each marker $m_{e'}$ of strategies of lower priorities is assigned the value $s+1$.

At stage $s+1$, assume that $A_{0,s} \cup A_{1,s} = [0,s)$ and that f is defined for each pair over $[0,s)$. For each $x \in [0,s)$, set $f(x,s) = i$ for the unique i such that $x \in A_{i,s}$. If some strategy requires attention at stage $s+1$, take the least one and satisfy it. If no such requirement is found, set $A_{0,s+1} = A_{0,s} \cup \{s\}$ and $A_{1,s+1} = A_{1,s}$. Then go to the next stage. This ends the construction.

Each time a strategy acts, it changes the markers of strategies of lower priority, and is declared satisfied. Once a strategy is satisfied, only a strategy of higher priority can injure it. Therefore, each strategy acts finitely often and the markers stabilize. It follows that the A's also stabilize and that f is a stable function.

Claim. For every $x < y < z$, $f(x,y) = 1 \wedge f(y,z) = 1 \rightarrow f(x,z) = 1$

Proof. Suppose that $f(x,y) = 1$ and $f(y,z) = 1$ but $f(x,z) = 0$. By construction of f, $x \in A_{0,z}$, $x \in A_{1,y}$ and $y \in A_{1,z}$. Let $s \leq z$ be the last stage such that $x \in A_{1,s}$. Then at stage $s+1$, some strategy $\mathscr{R}_e$ receives attention and moves x to $A_{0,s+1}$ and therefore moves $[x,s]$ to $A_{0,s+1}$. In particular $y \in A_{0,s+1}$ since $y \in [x,s]$. Moreover, the strategies of lower priority have had their marker moved to $s+1$ and therefore will never move any element below s. Since $f(y,z) = 1$, then $y \in A_{1,z}$. In particular, some strategy $\mathscr{R}_i$ of higher priority moved y to $A_{1,t+1}$ at stage $t+1$ for some $t \in (s,z)$. Since $\mathscr{R}_i$ has a higher priority, $m_i \leq m_e$, and since y is moved to $A_{1,t+1}$, then so is $[m_i, y]$, and in particular $x \in A_{1,t+1}$ since $m_i \leq m_e \leq x \leq y$. This contradicts the maximality of s.

Claim. For every $e \in \omega$, $\mathscr{R}_e$ is satisfied.

Proof. By induction over the priority order. Let s_0 be a stage after which no strategy of higher priority will ever act. By construction, m_e will not change after stage s_0. If $\varphi_e(U,V)$ is essential, then $\varphi_e(R,S)$ holds for two sets $m_e < R < S$. Let $s = 1 + max(s_0, S)$. The strategy $\mathscr{R}_e$ will require attention at some stage before s, will receive attention, be satisfied and never be injured.

This last claim finishes the proof. $\qquad\qquad\square$

Corollary 25. SCAC *does not admit preservation of dependent hyperimmunity.*

Proof. Let $f : [\omega]^2 \rightarrow 2$ be the coloring of Theorem 24. By construction, the pair $\overline{A}_0, \overline{A}_1$ is dependently hyperimmune, where $A_i = \{x : \lim_s f(x,s) = i\}$. Let H be an infinite f-homogeneous set. In particular, $H \subseteq A_0$ or $H \subseteq A_1$. We claim that the pair $\overline{A}_0, \overline{A}_1$ is not dependently H-hyperimmune. The $\Sigma_1^{0,H}$ formula $\varphi(U,V)$ defined by $U \neq \emptyset \wedge V \neq \emptyset \wedge U \cup V \subseteq H$ is essential since H is infinite. However, if there is some $R \subseteq A_1$ and $S \subseteq A_0$ such that $\varphi(R,S)$ holds, then $H \cap A_0 \neq \emptyset$ and $H \cap A_1 \neq \emptyset$, contradicting the choice of H. Therefore $\overline{A}_0, \overline{A}_1$ is not dependently H-hyperimmune. Hirschfeldt and Shore [12] proved that SCAC is equivalent to stable semi-transitive Ramsey's theorem for pairs over RCA_0. Therefore SCAC does not admit preservation of dependent hyperimmunity. $\qquad\square$

We will now prove the positive preservation result.

Theorem 26. *For every $k \geq 2$, psRT_k^2 admits preservation of dep. hyperimmunity.*

Proof. The proof is done by induction over $k \geq 2$. Fix a pair of sets $A_0, A_1 \subseteq \omega$ dependently Z-hyperimmune for some set Z. Let $f : [\omega]^2 \to k$ be a Z-computable coloring and suppose that there is no infinite set H over which f avoids at least one color, and such that the pair A_0, A_1 is dependently $H \oplus Z$-hyperimmune, as otherwise, we are done by induction hypothesis. We will build k infinite sets $G_0, \ldots, G_{k-1}$ such that G_i is pseudo-homogeneous for f with color i for each $i < k$ and such that A_0, A_1 is dependently $G_i \oplus Z$-hyperimmune for some $i < k$. The sets $G_0, \ldots, G_{k-1}$ are built by a variant of Mathias forcing $(F_0, \ldots, F_{k-1}, X)$ such that

(i) $F_i \cup \{x\}$ is pseudo-homogeneous for f with color i for each $x \in X$
(ii) X is an infinite set such that A_0, A_1 is dependently $X \oplus Z$-hyperimmune

A condition $d = (H_0, \ldots, H_{k-1}, Y)$ *extends* $c = (F_0, \ldots, F_{k-1}, X)$ (written $d \leq c$) if (H_i, Y) Mathias extends (F_i, X) for each $i < k$. A tuple of sets $G_0, \ldots, G_{k-1}$ *satisfies* c if for every $n \in \omega$, there is an extension $d = (H_0, \ldots, H_{k-1}, Y)$ of c such that $G_i \restriction n \subseteq H_i$ for each $i < k$. Informally, $G_0, \ldots, G_{k-1}$ satisfy c if the sets are generated by a decreasing sequence of conditions extending c. In particular, G_i is pseudo-homogeneous for f with color i and satisfies the Mathias condition (F_i, X). The first lemma shows that every sufficiently generic filter yields a k-tuple of infinite sets.

Lemma 27. *For every condition $c = (F_0, \ldots, F_{k-1}, X)$ and every $i < k$, there is an extension $d = (H_0, \ldots, H_{k-1}, Y)$ of c such that $|H_i| > |F_i|$.*

Proof. Fix c and $i < k$. If for every $x \in X$ and for all but finitely many $y \in X$, $f(x, y) \neq i$, then we could X-computably thin out the set X to obtain an infinite set H over which f avoids at least one color, contradicting our initial assumption. Therefore there must be some $x \in X$ such that the set $Y = \{y \in X : y > x \wedge f(x, y) = i\}$ is infinite. The condition $d = (F_0, \ldots, F_{i-1}, F_i \cup \{x\}, F_{i+1}, \ldots, F_{k-1}, Y)$ is the desired extension of c.

Fix an enumeration $\varphi_0(G, U, V), \varphi_1(G, U, V), \ldots$ of all $\Sigma_1^{0,Z}$ formulas. We want to satisfy the following requirements for each $e_0, \ldots, e_{k-1} \in \omega$:

$$\mathscr{R}_{\vec{e}} : \quad \mathscr{R}_{e_0}^{G_0} \quad \vee \quad \ldots \quad \vee \quad \mathscr{R}_{e_{k-1}}^{G_{k-1}}$$

where $\mathscr{R}_e^G$ is the requirement "$\varphi_e(G, U, V)$ essential $\to \varphi_e(G, R, S)$ for some $R \subseteq \overline{A}_0$ and $S \subseteq \overline{A}_1$". We say that a condition c *forces* $\mathscr{R}_{\vec{e}}$ if $\mathscr{R}_{\vec{e}}$ holds for every k-tuple of sets satisfying c. Note that the notion of satisfaction has a precise meaning given above.

Lemma 28. *For every condition c and every k-tuple of indices $e_0, \ldots, e_{k-1} \in \omega$, there is an extension d of c forcing $\mathscr{R}_{\vec{e}}$.*

Proof. Fix a condition $c = (F_0, \ldots, F_{k-1}, X)$. Let $\psi(U, V)$ be the $\Sigma_1^{0,X \oplus Z}$ formula which holds if there is a k-tuple of sets $E_0, \ldots, E_{k-1} \subseteq X$ and a $z \in X$ such that for each $i < k$,

(i) $z > max(E_i)$

(ii) $F_i \cup E_i \cup \{z\}$ is pseudo-homogeneous for color i.

(iii) $\varphi_{e_i}(F_i \cup E_i, U_i, V_i)$ holds for some $U_i \subseteq U$ and $V_i \subseteq V$

Suppose that c does not force $\mathscr{R}_{\vec{e}}$, otherwise we are done.

We claim that ψ is essential. Since c does not force $\mathscr{R}_{\vec{e}}$, there is a k-tuple of infinite sets $G_0, \ldots, G_{k-1}$ satisfying c and such that $\varphi_{e_i}(G_i, U, V)$ is essential for each $i < k$. Fix some $x \in \omega$. By definition of being essential, there are some finite sets $R_0, \ldots, R_{k-1} > x$ such that for every $y \in \omega$, there are finite sets $S_0, \ldots, S_{k-1} > y$ such that $\varphi_{e_i}(G_i, R_i, S_i)$ holds for each $i < k$. Let $R = \bigcup R_i$ and fix some $y \in \omega$. There are finite sets $S_0, \ldots, S_{k-1} > y$ such that $\varphi_{e_i}(G_i, R_i, S_i)$ holds for each $i < k$. Let $S = \bigcup S_i$. By continuity, there are finite sets $E_0, \ldots, E_{k-1}$ such that $G_i \restriction max(E_i) = F_i \cup E_i$ and $\varphi_{e_i}(F_i \cup E_i, R_i, S_i)$ holds for each $i < k$. By our precise definition of satisfaction, we can even assume without loss of generality that $(F_0 \cup E_0, \ldots, F_{k-1} \cup E_{k-1}, Y)$ is a valid extension of c for some infinite set $Y \subseteq X$. Let $z \in Y$. In particular, by the definition of being a condition extending c, $z \in X$, $z > max(E_0, \ldots, E_{k-1})$ and $F_i \cup E_i \cup \{z\}$ is pseudo-homogeneous for color i for each $i < k$. Therefore $\psi(R, S)$ holds, as witnessed by $E_0, \ldots, E_{k-1}$ and z. Thus $\psi(R, S)$ is essential.

Since A_0, A_1 is dependently $X \oplus Z$-hyperimmune, then $\psi(R, S)$ holds for some $R \subseteq \overline{A_0}$ and some $S \subseteq \overline{A_1}$. Let $E_0, \ldots, E_{k-1} \subseteq X$ be the k-tuple of sets and $z \in X$ be the integer witnessing $\psi(R, S)$. Let $i < k$ be such that the set $Y = \{w \in X \setminus [0, max(E_i)] : f(z, w) = i\}$ is infinite. The condition $d = (F_0, \ldots, F_{i-1}, F_i \cup E_i \cup \{z\}, F_{i+1}, \ldots, F_{k-1}, Y)$ is a valid extension of c forcing $\mathscr{R}_{\vec{e}}$.

Let $\mathscr{F} = \{c_0, c_1, \ldots\}$ be a sufficiently generic filter containing $(\emptyset, \ldots, \emptyset, \omega)$, where $c_s = (F_{0,s}, \ldots, F_{k-1,s}, X_s)$. The filter $\mathscr{F}$ yields a k-tuple of sets $G_0, \ldots, G_{k-1}$ defined by $G_i = \bigcup_s F_{i,s}$. By construction, $G_0, \ldots, G_{k-1}$ satisfies every condition in $\mathscr{F}$. By Lemma 27, the set G_i is infinite for each $i < k$ and by Lemma 28, the pair A_0, A_1 is dependently $G_i \oplus Z$-hyperimmune for some $i < k$. $\square$

Theorem 29. *Fix some set Z and a pair of sets A_0, A_1 dependently Z-hyperimmune. If Y is sufficiently random relative to Z, then the pair A_0, A_1 is dependently $Y \oplus Z$-hyperimmune.*

Proof. It suffices to prove that for every $\Sigma_1^{0,Z}$ formula $\varphi(G, U, V)$ and every $i \in \omega$, the following class is Lebesgue null.

$$\mathscr{S} = \{X : [\varphi(X, U, V) \text{ is essential }] \wedge (\forall R, S \subseteq_{\mathrm{fin}} \omega)\varphi(X, R, S) \rightarrow R \not\subseteq \overline{A_0} \vee S \not\subseteq \overline{A_1}\}$$

Suppose it is not the case. There exists $\sigma \in 2^{<\omega}$ such that

$$\mu\{X \in \mathscr{S} : \sigma \prec X\} > 0.8 \cdot 2^{-|\sigma|}$$

Define

$$\psi(U, V) = [\mu\{X \succ \sigma : (\exists \tilde{U} \subseteq U)(\exists \tilde{V} \subseteq V)\varphi(X, \tilde{U}, \tilde{V})\} > 0.6 \cdot 2^{-|\sigma|}]$$

By compactness, the formula $\psi(U, V)$ is $\Sigma_1^{0,Z}$.

Claim. $\psi(U, V)$ is essential.

Proof. Suppose it is not. Then, there exists some $x \in \omega$, such that for every $n \in \omega$, there is some $y_n \in \omega$ such that $\psi([x, n], [y_n, +\infty))$ does not hold. Let $\mathscr{P}(X, n, y_n)$ be the formula
$$(\forall \tilde{U} \subseteq [x, n])(\forall \tilde{V} \subseteq [y_n, +\infty)) \neg \varphi(X, \tilde{U}, \tilde{V})$$
Unfolding the definition of $\neg \psi([x, n], [y_n, +\infty))$,
$$\mu\{X \succ \sigma : \mathscr{P}(X, n, y_n)\} > 0.4 \cdot 2^{-|\sigma|}$$
Then, by Fatou's lemma,
$$\mu\{X \succ \sigma : (\exists^\infty n) \mathscr{P}(X, n, y_n)\} > 0.2 \cdot 2^{-|\sigma|}$$
Since whenever $\mathscr{P}(X, n, y_n)$ holds, so does $\mathscr{P}(X, n-1, y_n)$,
$$\mu\{X \succ \sigma : (\forall n)(\exists y) \mathscr{P}(X, n, y)\} > 0.2 \cdot 2^{-|\sigma|}$$
Therefore
$$\mu\{X \succ \sigma : \varphi(X, U, V) \text{ is essential }\} \leq 0.8 \cdot 2^{-|\sigma|}$$
Contradicting our assumption. This finishes the lemma.

By our claim and by dependent Z-hyperimmunity of A_0, A_1, there exists some finite sets $R \subseteq \bar{A}_0$ and $S \subseteq \bar{A}_1$ such that $\psi(R, S)$ holds. For every R, S such that $\psi(R, S)$ holds, there exists some $X \in \mathscr{S}$ and some $\tilde{R} \subseteq R$ and $\tilde{S} \subseteq S$ such that $\varphi(X, \tilde{R}, \tilde{S})$ holds. By definition of $X \in \mathscr{S}$, $\tilde{R} \not\subseteq \bar{A}_0$ or $\tilde{S} \not\subseteq \bar{A}_1$ and therefore either $R \not\subseteq \bar{A}_0$ or $S \not\subseteq \bar{A}_1$. Contradiction. $\square$

Corollary 30. WWKL *admits preservation of dependent hyperimmunity.*

Proof. Let Z be a set and A_0, A_1 be a pair of dependently Z-hyperimmune sets. Fix a Z-computable tree of positive measure $T \subseteq 2^{<\omega}$. By Theorem 29, the pair A_0, A_1 is dependently $Y \oplus Z$-hyperimmune for some Martin-Löf random Y relative to Z. By Kučera [16], Y is, up to finite prefix, a path through T. $\square$

Corollary 31. *For every $k \geq 2$,* $\mathsf{RCA}_0 \wedge \mathsf{psRT}_k^2 \wedge \mathsf{WWKL} \nvdash \mathsf{SCAC}$.

Proof. Immediate by Theorem 26, Corollary 30, and Corollary 25. $\square$

Corollary 32. $\mathsf{RCA}_0 \wedge \mathsf{ADS} \wedge \mathsf{WWKL} \nvdash \mathsf{SCAC}$

Proof. Immediate by the previous corollary and Theorem 24 of [19]. $\square$

Whenever requiring the sets A_0 and A_1 to be co-c.e., we recover the standard notion of hyperimmunity. Therefore, the restriction of the preservation of dependent hyperimmunity to co-c.e. sets is not a good computability-theoretic property to distinguish consequences of Ramsey's theorem for pairs.

Lemma 33. *Fix two sets A_0, A_1 such that A_0 is X-co-c.e. The pair A_0, A_1 is dependently X-hyperimmune iff A_0 and A_1 are X-hyperimmune.*

Proof. We first show that if A_0 and A_1 are dependently X-hyperimmune then both A_0 and A_1 are X-hyperimmune. Let $F_0, F_1, \ldots$ be a X-c.e. array. Let $\varphi(U, V)$ be the $\Sigma_1^{0,X}$ formula which holds if $U = F_i$ for some $i \in \omega$. The formula $\varphi(U, V)$ is essential, therefore there $\varphi(R, S)$ holds for some finite set $R \subseteq \bar{A}_0$ and $S \subseteq \bar{A}_1$. In particular, $R = F_i$ for some $i \in \omega$, therefore $F_i \subseteq \bar{A}_0$ and A_0 is hyperimmune. Similarly, the $\Sigma_1^{0,X}$ formula $\psi(U, V)$ which holds if $V = F_i$ for some $i \in \omega$ witnesses that A_1 is hyperimmune.

We now prove that if A_0 and A_1 are X-hyperimmune and A_0 is X-co-c.e., then the pair A_0, A_1 is dependently X-hyperimmune. Let $\varphi(U, V)$ be an essential $\Sigma_1^{0,X}$ formula. Define an X-c.e. sequence of sets $F_0 < F_1 < \ldots$ such that for every $i \in \omega$, there is some $R < F_i$ such that $\varphi(R, F_i)$ holds and $R \subseteq \bar{A}_0$. First, notice that the sequence is X-c.e. since A_0 is X-co-c.e. Second, we claim that the sequence is infinite. To see this, define an X-c.e. array $E_0 < E_1 < \ldots$ such that for every $i \in \omega$, there is some finite set $S > E_i$ such that $\psi(E_i, S)$ holds. The array is infinite since $\psi(U, V)$ is essential. Since A_0 is X-hyperimmune, there are infinitely many i's such that $E_i \subseteq \bar{A}_0$. Last, by X-hyperimmunity of A_1, there is some $i \in \omega$ such that $F_i \subseteq \bar{A}_1$. By definition of F_i, there is some $R \subseteq \bar{A}_0$ such that $\varphi(R, F_i)$ holds. □

Corollary 34. RT_2^2 *admits preservation of dependent hyperimmunity for co-c.e. sets.*

6 Weakly stable partial orders

In their seminal paper [4], Cholak, Jockusch and Slaman had the idea to split Ramsey's theorem for pairs into two simpler statements, namely, stable Ramsey's theorem for pairs (SRT_2^2) and cohesiveness (COH), in order to get more insights about the strength of RT_2^2 by manipulating SRT_2^2 and COH independently. Hirschfeldt and Shore [12] applied the same idea to their statements about linear and partial orders, and introduced the corresponding notions of stability. In the case of partial orders, there are however two possible notions of stability.

Given a partial order $(P, \leq_P)$, we say that $x \in P$ is *small*, *large* or *isolated* if for all but finitely many $y \in P$, $x \leq_P y$, $x \geq_P y$, or $x |_P y$, respectively. We write $S^*(P)$, $L^*(P)$ and $I^*(P)$ for the set of small, large and isolated elements of P, respectively. A partial order is *weakly stable* if every element is either small, large, or isolated, that is, $P = S^*(P) \cup L^*(P) \cup I^*(P)$. A partial order is *stable* if every element is small or isolated, or if every element is large or isolated, that is, $P = S^*(P) \cup I^*(P)$ or $P = L^*(P) \cup I^*(P)$. We let SCAC and WSCAC be the restriction of CAC to stable and weakly stable partial orders, respectively.

The notion of stable partial order was introduced by Hirschfeldt and Shore. They proved that ADS is equivalent to the statement "Every partial order has a stable suborder", showing therefore that $\mathsf{RCA}_0 \vdash \mathsf{CAC} \leftrightarrow \mathsf{ADS} \wedge \mathsf{SCAC}$. Hirschfeldt and Shore noticed that SCAC was not the most immediate notion of stability, but justified their choice by proving an equivalence between SCAC and the restriction of SRT_2^2 to semi-transitive colorings. Furthermore, Jockusch et al. [15] proved that $\mathsf{RCA}_0 \vdash \mathsf{SCAC} \leftrightarrow \mathsf{WSCAC}$, showing therefore that the choice of either notion had no impact to the strength of the statement in reverse mathematics. However,

the implication SCAC $\to$ WSCAC involved two applications of SCAC, and Astor et al. [1] proved that those two applications were necessary, by showing that WSCAC $\not\leq_c$ SCAC. Their proof uses a very involved notion of forcing building an instance of WSCAC and solutions to SCAC simultaneously. In this section, we give a simpler proof formulated in terms of immunity, and furthermore show that the instance of WSCAC can be chosen to be computable.

Definition 35 (Combined immunity). *A formula $\varphi(u,v)$ is* combinedly essential *if for every $x \in \omega$, there are some $r,s > x$ such that $r \neq s$ and $\varphi(r,s)$ holds. A pair of sets $A_0, A_1 \subseteq \omega$ is* combinedly X-immune *if for every combinedly essential $\Sigma_1^{0,X}$ formula $\varphi(u,v)$, $\varphi(r,s)$ holds for some $r \in \overline{A}_0$ and $s \in \overline{A}_1$.*

In particular, if A_0 and A_1 are combinedly X-immune, then so are A_1 and A_0. Moreover, they are both X-immune. Note that the notion of combined immunity differs from dependent immunity by the alternation of quantifiers in the definition of essentiality.

Theorem 36. *There is a computable weakly stable partial order $(P, \leq_P)$ such that $S^*(P) \cup L^*(P)$ and $I^*(P)$ are both hyperimmune, and are combinedly immune.*

Proof. Fix an enumeration $\varphi_0(U), \varphi_1(U), \dots$ of all Σ_1^0 formulas where U is a finite coded set parameter, and an enumeration $\psi_0(u,v), \psi_1(u,v), \dots$ of all Σ_1^0 formulas where u and v are integer parameters. The construction of the partial order $(P, \leq_P)$ is done by a finite injury priority argument with a movable marker procedure. Recall that a formula $\varphi(U)$ where U is a finite coded set is *essential* if for every $x \in \omega$, there is some finite set $R > x$ such that $\varphi(R)$ holds. The following schemes of requirements ensure that $I^*(P)$ and $S^*(P) \cup L^*(P)$ will both be hyperimmune.

$$\mathscr{R}_e : \varphi_e(U) \text{ essential} \to (\exists R \subseteq_{fin} S^*(P) \cup L^*(P))\varphi_e(R)$$

$$\mathscr{S}_e : \varphi_e(U) \text{ essential} \to (\exists R \subseteq_{fin} I^*(P))\varphi_e(R)$$

The following scheme of requirements ensures that $S^*(P) \cup L^*(P)$ and $I^*(P)$ are combinedly immune.

$$\mathscr{T}_e : \psi_e(u,v) \text{ combinedly essential} \to (\exists r \in I^*(P))(\exists s \in S^*(P) \cup L^*(P))\psi_e(r,s)$$

The requirements are given an interleaved priority ordering. We proceed by stages, maintaining three sets S, L and I, which represent $S^*(P)$, $L^*(P)$ and $I^*(P)$, respectively. At stage 0, $S_0 = L_0 = \emptyset$ and $I_0 = \{0\}$ and $\leq_P$ is nowhere defined. Moreover, to each group of requirements $\mathscr{R}_e, \mathscr{S}_e, \mathscr{T}_e$, we associate a marker m_e, initialized to 0.

A strategy for $\mathscr{R}_e$ *requires attention at stage $s+1$* if $\varphi_e(R)$ holds for some $R \subseteq [m_e, s]$. The strategy sets $S_{s+1} = S_s \cup [m_e, s]$, $L_{s+1} = L_s \smallsetminus [m_e, s]$ and $I_{s+1} = I_s \smallsetminus [m_e, s]$.

A strategy for $\mathscr{S}_e$ *requires attention at stage $s+1$* if $\varphi_e(R)$ holds for some $R \subseteq [m_e, s]$. The strategy sets $S_{s+1} = S_s \smallsetminus [m_e, s]$, $L_{s+1} = L_s \smallsetminus [m_e, s]$, and $I_{s+1} = I_s \cup [m_e, s]$.

A strategy for $\mathscr{T}_e$ *requires attention at stage $s+1$* if $\varphi_e(u,v)$ holds for some $u, v \in [m_e, s]$ such that $u \neq v$. Let $\{v\} \uparrow = \{x \in [m_e, s] : v \leq_P x\}$ and $\{v\} \downarrow = \{x \in [m_e, s] :$

$v \geq_P x\}$. If $u \leq_P v$, then the strategy sets $S_{s+1} = S_s \smallsetminus [m_e, s]$, $L_{s+1} = (L_s \smallsetminus [m_e, s]) \cup \{v\} \uparrow$, and $I_{s+1} = I_s \cup ([m_e, s] \smallsetminus \{v\} \uparrow)$. If $u \not\leq_P v$, then the strategy sets $S_{s+1} = (S_s \smallsetminus [m_e, s]) \cup \{v\} \downarrow$, $L_{s+1} = L_s \smallsetminus [m_e, s]$, and $I_{s+1} = I_s \cup ([m_e, s] \smallsetminus \{v\} \downarrow)$.

At stage $s + 1$, assume that $S_s \cup L_s \cup I_s = [0, s]$ and that $\leq_P$ is defined for each pair over $[0, s)$. For each $x \in [0, s)$, set $x \leq_P s$ if $x \in S_s$, $x \geq_P s$ if $x \in L_s$, and $x |_P s$ if I_s. If some strategy $\mathscr{R}_e$, $\mathscr{S}_e$ or $\mathscr{T}_e$ requires attention at stage $s + 1$, take the least such one and execute it. Then, declare the strategy satisfied, declare all the strategies of lower priority unsatisfied, and set $m_i = s + 1$ for every $i \geq e$. If no strategy requires attention, then add $\{s\}$ to I_{s+1} and go to the next stage. This ends the construction.

Each time a strategy acts, it changes the marker of all strategies of lower priority, and is declared satisfied. Once a strategy is satisfied, only a strategy of higher priority can injure it. Therefore, each strategy acts finitely often, and the markers stabilize. It follows that relation $\leq_P$ is weakly stable.

Claim. The relation $\leq_P$ is transitive.

Proof. Suppose that there are three elements $x, y <_\mathbb{N} z$ such that $x \leq_P y \leq_P z \leq_P x$. We have two cases. In the first case, $x <_\mathbb{N} y$. Then, by construction, $x \in S_y \cap L_z$ and $y \in S_z$. Then, x must have been moved to L at a stage s between stage y and stage z, otherwise we would have $x \geq_P y$ or $x \notin L_z$. Let s_0 be the last such stage. When x is moved to L at stage s_0, it is because of some requirement $\mathscr{R}_e$, $\mathscr{S}_e$ or $\mathscr{T}_e$ such that $m_e \leq_\mathbb{N} x$. But then, after this, m_i is moved to a value greater than $s_0 \geq y$ for every $i \geq e$. By construction, when x is moved to L after stage y, then so is $\{x\} \uparrow$, and in particular so is y. Therefore, if $y \in S_z$, it must have been moved out of L at some later stage $s_1 \geq s_0$, and by a strategy of higher priority $\mathscr{R}_i$, $\mathscr{S}_i$ or $\mathscr{T}_i$, for some $i < e$. By construction, at stage s_0, $m_i \leq_\mathbb{N} m_e \leq_\mathbb{N} x$. We claim that $m_i \leq_\mathbb{N} x$ at stage s_1. If not, then the value of m_i must have been changed at a stage between s_0 and s_1, but then by construction, m_i has been moved to a value greater than $s_0 \geq y$. Since $\mathscr{R}_i$, $\mathscr{S}_i$ or $\mathscr{T}_i$ can change only values greater than m_i, and since y have been changed, we obtain a contradiction. The movement of y can be due to $\mathscr{R}_i$, in which case $x \in [m_i, s_1] \subseteq S_{s_1}$ since $m_i \leq_\mathbb{N} x \leq_\mathbb{N} s_1$, or it can be due to $\mathscr{T}_i$, in which case $y \in \{v\} \downarrow$ for some v, and hence $x \in \{y\} \downarrow \subseteq \{v\} \downarrow \subseteq S_{s_1}$. In both cases, $x \in S_{s_1}$. By maximality of s_0, x does not enter again in L before stage z, contradicting $x \in L_z$. The case where $x >_\mathbb{N} y$ is treated similarly.

Claim. For every $e \in \omega$, $\mathscr{R}_e$, $\mathscr{S}_e$ and $\mathscr{T}_e$ are satisfied.

Proof. By induction over the priority order. Fix some e, and let s_0 be a stage after m_e does not change any more. Suppose for contradiction that $\mathscr{R}_e$ is not satisfied at stage s_0. Then $\varphi_e(U)$ is essential, so $\varphi_e(R)$ holds for some set $R > m_e$. The strategy $\mathscr{R}_e$ will require attention at some stage before $\max(s_0, R)$, and will receive attention since no strategy of higher priority acts after stage s_0. Then it will act, and m_e will be moved to a value greater than s_0, contradiction. The cases of $\mathscr{S}_e$ and $\mathscr{T}_e$ is similar.

This last claim finishes the proof. □

Theorem 37. *For every computable stable partial order $(P, \leq_P)$ and for every pair of sets A_0, A_1 which are both hyperimmune, and are combinedly immune, there is an infinite chain or antichain H such that A_0 and A_1 are both H-immune.*

Proof. Fix a computable partial stable partial order $(P, \leq_P)$, and assume that $P = S^*(P) \cup I^*(P)$. The other case is symmetric. Assume that there is no chain or antichain H such that A_0 and A_1 are both H-immune, otherwise we are done. We will build two infinite sets G_0 and G_1, such that G_0 is an infinite ascending sequence, G_1 is an infinite antichain, and such that A_0 and A_1 are both G_i-immune for some $i < 2$.

The construction is done by a variant of Mathias forcing (F_0, F_1), where $F_0 \subseteq S^*(P)$ is a finite ascending sequence and $F_1 \subseteq I^*(P)$ is a finite antichain. Given a condition $c = (F_0, F_1)$, we let

$$X(c) = \{z \in \omega : (\forall x \in F_0)[x < z \wedge x <_P z] \wedge (\forall y \in F_1)[y < z \wedge y |_P z]\}$$

Note that the set $X(c)$ is cofinite. A condition $d = (E_0, E_1)$ *extends* $c = (F_0, F_1)$ if $(E_i, X(d))$ Mathias extends $(F_i, X(c))$ for each $i < 2$. A pair of sets G_0, G_1 *satisfies* a condition $c = (F_0, F_1)$ if G_0 is an infinite ascending sequence, G_1 is an infinite antichain, and G_i satisfies the Mathias condition $(F_i, X(c))$ for each $i < 2$.

Lemma 38. *For every condition $c = (F_0, F_1)$, there is an extension (E_0, E_1) of c such that $|E_i| > |F_i|$ for each $i < 2$.*

Proof. By assumption, $L^*(P) \cap X(c)$ and $I^*(P) \cap X(c)$ are both infinite, otherwise there would be a computable infinite chain or antichain. Take any $x \in L^*(P) \cap X(c)$ and $y \in I^*(P) \cap X(c)$. The condition $(F_0 \cup \{x\}, F_1 \cup \{y\})$ is the desired extension. $\quad\square$

A condition c *forces* a formula $\varphi(G_0, G_1)$ if $\varphi(G_0, G_1)$ holds for every pair of sets G_0, G_1 satisfying c. We want to satisfy the disjunctive requirement $\mathscr{R}_{e_0, e_1, i_0, i_1} = \mathscr{R}_{e_0, i_0}^{G_0} \vee \mathscr{R}_{e_1, i_1}^{G_1}$ for each $e_0, e_1 \in \omega$ and $i_0, i_1 \in \{0, 1\}$, where

$$\mathscr{R}_{e, i}^{G} : \Phi_e^G \text{ is not an infinite subset of } A_i$$

Note that to obtain the desired property, the pair (i_0, i_1) must range over $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$. For this, we are going to use the fact that (A_0, A_0), (A_0, A_1), (A_1, A_0), and (A_1, A_1) are all combinedly immune, which is a consequence of the facts that (A_0, A_1) are combinedly immune and that both A_0 and A_1 are hyperimmune.

Lemma 39. *For every condition c, every pair of indices $e_0, e_1 \in \omega$ and every $i_0, i_1 \in \{0, 1\}$, there is an extension d of c forcing $\mathscr{R}_{e_0, e_1, i_0, i_1}$.*

Proof. Fix a condition $c = (F_0, F_1)$. A *split pair* is a pair of finite sets $E_0, E_1 \subseteq X(c)$ such that $F_0 \cup E_0$ is a finite ascending sequence, $F_1 \cup E_1$ is a finite antichain, and $\max E_0 \leq_P x$ for each $x \in E_1$. Let $\varphi(u, v)$ be the Σ_1^0 formula which holds if there is a split pair E_0, E_1, such that $\Phi_{e_0}^{F_0 \cup E_0}(u) \downarrow$ and $\Phi_{e_1}^{F_1 \cup E_1}(v) \downarrow$. We have two cases.

- Case 1: the formula $\varphi(u, v)$ is not combinedly essential, say with witness x. If there is a pair of infinite sets G_0, G_1 satisfying c and some $r > x$ such that $\Phi_{e_0}^{G_0}(r) \downarrow$, then let $E_0 \subseteq G_0$ be such that $F_0 \cup E_0$ is an initial segment of G_0 for which $\Phi_{e_0}^{F_0 \cup E_0}(r) \downarrow$. In particular, $E_0 \subseteq L^*(P) \cap X(c)$ is an finite increasing sequence, so the condition $d = (F_0 \cup E_0, F_1)$ is an extension forcing $\mathscr{R}_{e_1, i_1}^{G_1}$, hence forcing $\mathscr{R}_{e_0, e_1, i_0, i_1}$. If there is no such pair of infinite sets G_0, G_1, then the condition c already forces $\mathscr{R}_{e_0, i_0}^{G_0}$, hence $\mathscr{R}_{e_0, e_1, i_0, i_1}$.

- Case 2: the formula $\varphi(u,v)$ is combinedly essential. Note that since the pair A_0, A_1 is combinedly immune, so is the pair A_1, A_0. Moreover, since for each $i \in \{0,1\}$, A_i is hyperimmune, the pair A_i, A_i is combinedly immune. Therefore, for every $i_0, i_1 \in \{0,1\}$, the pair A_{i_0}, A_{i_1} is combinedly immune. In particular, $\varphi(r,s)$ holds for some $r \in \overline{A}_{i_0}$ and $s \in \overline{A}_{i_1}$.

  Unfolding the definition of $\varphi(r,s)$, there is a split pair E_0, E_1 such that $\Phi_{e_0}^{F_0 \cup E_0} \cap \overline{A}_{i_0} \neq \emptyset$ and $\Phi_{e_1}^{F_1 \cup E_1} \cap \overline{A}_{i_1} \neq \emptyset$. Since $\max E_0 <_P x$ for each $x \in E_1$, either $E_0 \subseteq L^*(P)$, or $E_1 \subseteq I^*(P)$. Therefore, either $d_0 = (F_0 \cup E_0, F_1)$, or $d_1 = (F_0, F_1 \cup E_1)$ is a valid extension of c. In particular, d_0 forces $\mathscr{R}_{e_0,i_0}^{G_0}$ and d_1 forces $\mathscr{R}_{e_1,i_1}^{G_1}$. In both case, there is an extension of c forcing $\mathscr{R}_{e_0,e_1,i_0,i_1}$.

Let $\mathscr{F} = \{c_0, c_1, \ldots\}$ be a sufficiently generic filter containing $(\emptyset, \emptyset)$, where $c_s = (F_{0,s}, F_{1,s})$. The filter $\mathscr{F}$ yields a pair of sets G_0, G_1 defined by $G_i = \bigcup_s F_{i,s}$. By Lemma 38, the sets G_0 and G_1 are both infinite, and by Lemma 39, the sets A_0 and A_1 are both G_i-immune for some $i < 2$. This completes the proof of Theorem 37. $\quad\square$

Corollary 40. WSCAC $\not\leq_c$ SCAC.

Proof. By Theorem 36, there is a computable, weakly stable partial order $(P, \leq_P)$ such that $S^*(P) \cup L^*(P)$ and $I^*(P)$ are both hyperimmune, and are combinedly immune. By Theorem 37, for every computable stable partial order $(Q, \leq_Q)$, there is an infinite Q-chain or Q-antichain H such that $S^*(P) \cup L^*(P)$ and $I^*(P)$ are both H-immune, and therefore such that H does not compute an infinite P-chain or P-antichain.

Accordingly, we say that a linear order $(P, \leq_P)$ is *stable* if it is of order type $\omega + \omega^*$, that is, if $P = S^*(P) \cup L^*(P)$. We let SADS be the restriction of ADS to stable colorings. Tennenbaum (see Downey [6]) constructed a computable linear order of order type $\omega + \omega^*$ with no infinite computable ascending or descending sequence. Downey noticed that the construction could be modified so that the ω and ω^* part are both hyperimmune. We now show that this is the case for *every* computable instance of SADS with no computable solution. It follows that every such instance of SADS is a witness that the Erdős-Moser theorem (EM) does not imply SADS over RCA$_0$, since the former has been proven to admit preservation of hyperimmunity (see [17,20]).

Lemma 41. *For every computable linear order $(P, \leq_P)$ of order type $\omega + \omega^*$ with no computable infinite ascending or descending sequence, $S^*(P)$ and $L^*(P)$ are both hyperimmune.*

Proof. We first prove that $S^*(P)$ is hyperimmune. Let $F_0, F_1, \ldots$ be an array tracing $S^*(P)$. Then, the set $\{\min_P F_i : i \in \omega\}$ is an infinite $\vec{F}$-computable subset of $S^*(P)$, and therefore the array is not computable. It follows that $S^*(P)$ is not traced by any computable array. The case of $L^*(P)$ holds by symmetry.

7 Partial orders with compactness

The framework of preservation of computability-theoretic properties enables one, among other things, to separate compound statements in reverse mathematics by

analyzing the preservation of such properties by each statement separately. For example, Wang [25] separated $COH + WKL + EM$ from SADS over RCA_0 using the preservation of proper Δ_2^0 definitions. In a previous version of this paper, we asked the following question.

Question 42. Does $ADS + WKL$ imply any of RT_2^2 or CAC over RCA_0?

The statements ADS and WKL have both known not to imply RT_2^2 over RCA_0, but for very different reasons. By proving in Section 3 that CAC admits preservation of c.b-immunity, we showed that CAC, and *a fortiori* ADS, does not imply any notion of compactness. On the other hand, RT_2^2 implies the diagonally non-computable principle (DNC), which is equivalent to a very weak form of compactness, namely, the Ramsey-type weak weak König's lemma (see [2,8]).

Weak König's lemma, as for him, does not imply RT_2^2 over RCA_0 for various reasons. By the low basis theorem [14], one can build a model of WKL containing only low sets, while Jockusch [13] constructed a computable instance of RT_2^2 with no Δ_2^0 solution. One can also separate WKL from RT_2^2 thanks the the hyperimmune-free basis theorem [14]. Indeed, RT_2^2 has a computable instance whose solutions are of hyperimmune degree. Both separations cannot be adapted to separate $WKL + ADS$ from RT_2^2 since by Hirshchfeldt and Shore [12], $RCA_0 \vdash ADS \rightarrow COH$ and therefore ADS has a computable instance with no low solution and whose solutions are of hyperimmune degree.

The question was recently answered negatively by Towsner [24], who developed an involved technique for separating statements about partial orders in presence of weak König's lemma.

Theorem 43 (Towsner [24]). $ADS + WKL$ *does not imply* CAC*, and* $CAC + WKL$ *does not imply* RT_2^2 *over* RCA_0.

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