Custom properties 6

Prerequisites: Chapters 2, 3 and 5

The classical study of computability theory puts the emphasis on some concepts such as hyperimmunity, PA degrees, or the arithmetic hierarchy. These notions induce invariant properties like preservation of hyperimmunity, PA avoidance, or low$_n$ness, enabling to separate second-order statements in reverse mathematics. However, the diversity of second-order statements makes it impossible to always separate them with classical notions.

In this chapter, we explain how to design custom computability-theoretic properties to separate two mathematical problems. As it turns out, their design is once again driven by the definability and combinatorial properties of their corresponding forcing questions. The main ideas are presented in this chapter through the study of three important statements: the Erdős-Moser theorem (EM), the Ascending Descending Sequence principle (ADS) and the Chain-AntiChain principle (CAC).

6.1 Separation framework

Consider two Π^1_2 problems P and Q. In order to separate P from Q over RCA_0, one needs to build a model $\mathcal{M} \models \mathsf{RCA}_0 + \mathsf{P}$ containing an instance X_Q, but such that $\mathcal{M}$ contains no Q-solution to X_Q. The model $\mathcal{M}$ is usually built as a limit of a countable increasing sequence $\mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \dots$ of Turing ideals as follows. First, construct a Q-instance X_Q with no X_Q-computable solution, and let $\mathcal{M}_0 = \{Y \in 2^\mathbb{N} : Y \leq_T X_\mathsf{Q}\}$. Then, assuming $\mathcal{M}_n$ is a Turing ideal of the form $\{Y \in 2^\mathbb{N} : Y \leq_T Z_n\}$[1] for some set Z_n, pick a P-instance X_P in $\mathcal{M}_n$ with no solution in $\mathcal{M}_n$, construct a solution Y_P to X_P, and let $\mathcal{M}_{n+1} = \{Y \in 2^\mathbb{N} : Y \leq_T Z_n \oplus Y_\mathsf{P}\}$. One furthermore wants to maintain the invariant that X_Q has no Q-solution in $\mathcal{M}_n$, so the difficulty is to build a solution Y_P to X_P such that X_Q has no $Z_n \oplus Y_\mathsf{P}$-computable solution, assuming it has no Z_n-computable solution. Usually, one needs to find a stronger invariant than just having no Z_n-computable solution. A class $\mathcal{W} \subseteq 2^\mathbb{N}$ is a *weakness property* if it is downward-closed under the Turing reduction.

1: Turing ideal of this form are called topped. A model of RCA_0 is *topped* if its corresponding Turing ideal is topped.

Definition 6.1.1. A problem P *preserves* a weakness property $\mathcal{W}$ if for every $Z \in \mathcal{W}$ and every Z-computable instance X, there is a solution Y to X such that $Z \oplus Y \in \mathcal{W}$. ◊

This previous definition generalizes many properties defined in the previous chapters. For instance, a problem P admits cone avoidance iff it preserves $\mathcal{W}_C = \{X \in 2^\mathbb{N} : C \not\leq_T X\}$ for every set C.[2]

2: Note that if C is computable, then $\mathcal{W}_C = \emptyset$, and then P vacuously preserves $\mathcal{W}_C$.

Exercise 6.1.2. Formulate PA avoidance (Definition 5.1.1) as a preservation of a family of weakness properties. ⋆

The following theorem gives the general construction underlying almost all the separation proofs over ω-models.

Theorem 6.1.3
Let P *be a* Π^1_2 *problem preserving a weakness property* $\mathcal{W}$*. Then for every set* $Z \in \mathcal{W}$*, there is an* ω*-model* $\mathcal{M}$ *of* $\mathsf{RCA}_0 + \mathsf{P}$ *such that* $\mathcal{M} \subseteq \mathcal{W}$ *and* $Z \in \mathcal{M}$*.*

Proof. We are going to define a countable sequence of Turing ideals $\mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \dots$, where $\mathcal{M}_n = \{Y \in 2^{\mathbb{N}} : Y \leq_T Z_n\}$, such that for all $n \in \mathbb{N}$,

(1) if $n = \langle a, b\rangle$ and X is the a-th P-instance of $\mathcal{M}_b$, then Z_{n+1} computes a P-solution to X;
(2) $Z_{n+1} \in \mathcal{W}$, or equivalently $\mathcal{M}_n \subseteq \mathcal{W}$.

First $Z_0 = Z$. Suppose we have defined $Z_n \in \mathcal{W}$ and say $n = \langle a, b\rangle$. Let X be the a-th P-instance of $\mathcal{M}_b$, Since P preserves $\mathcal{W}$, there is a solution Y to X such that $Y \oplus Z_n \in \mathcal{W}$. Let $Z_{n+1} = Z_n \oplus Y$. This completes the construction.

Let $\mathcal{M} = \bigcup_n \mathcal{M}_n = \{Y \in 2^{\mathbb{N}} : \exists n\ Y \leq_T Z_n\}$. By construction, the class $\mathcal{M}$ is a Turing ideal, thus $\mathcal{M} \models \mathsf{RCA}_0$. Moreover, by (1), every P-instance $X \in \mathcal{M}$ admits a solution in $\mathcal{M}$. By (2), $\mathcal{M} \subseteq \mathcal{W}$ and by construction, $Z \in \mathcal{M}$. ■

Corollary 6.1.4
Fix a weakness property $\mathcal{W}$*. Let* P *and* Q *be two* Π^1_2 *problems such that* P *preserves* $\mathcal{W}$ *but* Q *does not. Then* $\mathsf{RCA}_0 + \mathsf{P} \nvdash \mathsf{Q}$*.*

Proof. Since Q does not preserve $\mathcal{W}$, there is some $Z \in \mathcal{W}$ and some Z-computable instance X_{Q} of Q such that for every solution Y to X_{Q}, $Y \oplus X_{\mathsf{Q}} \notin \mathcal{W}$. Since P preserves $\mathcal{W}$, by Theorem 6.1.3, there is an ω-model $\mathcal{M}$ of $\mathsf{RCA}_0 + \mathsf{P}$ such that $\mathcal{M} \subseteq \mathcal{W}$ and $Z \in \mathcal{M}$. In particular, $X_{\mathsf{Q}} \in \mathcal{M}$, but $\mathcal{M}$ does not contain any Q-solution to X_{Q}, so $\mathcal{M} \not\models \mathsf{Q}$. ■

The purpose of this chapter is to emphasize the relation between the combinatorial features of the forcing question of a problem P and the invariant properties it preserves, and to learn through examples how to design a custom invariant property to separate two problems.

6.2 Immunity and variants

The early study of reverse mathematics has shown the emergence of an empirical structural phenomenon: the vast majority of ordinary theorems of mathematics, once formulated as second-order statements, are either provable over RCA_0, or provably equivalent over RCA_0 to one among four main systems of axioms, namely, WKL_0, ACA_0, ATR_0 and Π^1_1-CA_0.[3] These systems can be separated over ω-models using standard notions from computability theory or higher recursion theory. Thus, when considering two second-order statements, they are likely to be either equivalent over RCA_0, or to belong to two of the above-mentioned systems, and therefore separable using standard notions.

3: These systems are known as the "Big Five" (see Montalbán [40]).

Some exceptions exist to this structural phenomenon, mostly coming from Ramsey theory.[4] Overall, Ramsey's theory seeks to understand the inherent structure and order that can arise within large sets by investigating the existence of specific patterns, colorings, or configurations. In the setting of second-order arithmetic, statements from Ramsey theory assert the existence of infinite

4: One can often define "Ramsey-type" versions of standard problems, where a solution is an infinite number of bits of information of the original solution. For instance, the Ramsey-type weak König's lemma (RWKL) is a Ramsey-type version of weak König's lemma, stating the existence of an infinite set homogeneous for one of the path.

sets satisfying some property which is closed under subset. For instance, Ramsey's theorem states the existence, for every coloring $f : [\mathbb{N}]^n \to k$, of an infinite f-homogeneous set H, and every infinite subset $G \subseteq H$ is also f-homogeneous, hence also a solution. We shall therefore give a particular attention to statements such that the collection of solutions is closed under infinite subsets.

It follows that if Q is a statement from Ramsey theory and X is an instance with no computable solution, then every solution Y is immune.[5] Thus, when separating a Π^1_2 problem P from a Q over ω-models, one usually considers preservations of strong notions of immunity. Some of the invariant properties studied in previous chapters can already be formulated in terms of preservation of strong immunity.

5: Recall that an infinite set A is *immune* if it has no infinite computable subset, or equivalently if it has no infinite c.e. subset.

Hyperimmunity. As explained in Section 3.6, cone avoidance is equivalent to preservation of 1 hyperimmunity. In Chapter 2, hyperimmunity is defined in terms of domination of functions, but the original definition over sets is formulated as a strong variant of immunity.

> **Definition 6.2.1.** Let $D_0, D_1, \ldots$ be a canonical enumeration of all non-empty finite sets.[6] A *c.e. array*[7] is a collection of finite sets for the form $\{D_{f(n)} : n \in \mathbb{N}\}$ for some computable function $f : \mathbb{N} \to \mathbb{N}$, such that $\min D_{f(n)} > n$ for every $n \in \mathbb{N}$. An infinite set A is *hyperimmune* if for every c.e. array $\{D_{f(n)} : n \in \mathbb{N}\}$, there is some $n \in \mathbb{N}$ such that $A \cap D_{f(n)} = \emptyset$.

6: One can let D_n be such that $\sum_{x \in D_n} 2^x = n+1$, in other words, the binary representation of $n+1$ is seen as the characteristic function of D_n.

7: One usually requires a c.e. array to be made of pairwise disjoint sets rather than requiring that $\min D_{f(n)} > n$. Both definitions yield the same notion of hyperimmunity, but our formulation will be more convenient for merging c.e. arrays.

Intuitively, an infinite set A is hyperimmune if not only one cannot find an infinite subset of it, but one cannot even approximate an infinite subset by giving blocks of elements, each of them capturing an element of A. It is clear from the definition that if A is hyperimmune, then A is immune.

Exercise 6.2.2 (Kuznecov, Medvedev, Uspenskii). Recall that the *principal function* of an infinite set $A = \{x_0 < x_1 < \ldots\}$ is the function $p_A : \mathbb{N} \to \mathbb{N}$ defined by $p_A(n) = x_n$. Show that an infinite set A is hyperimmune iff its principal function p_A is hyperimmune, that is, is not dominated by any computable function. ★

Diagonal non-computability. Recall that a total function $f : \mathbb{N} \to \mathbb{N}$ is *diagonally non-computable* (DNC) if $f(e) \neq \Phi_e(e)$ for every $e \in \mathbb{N}$. The degrees computing DNC function admit many characterizations, and thus are arguably natural. By Proposition 5.7.2, a set X computes a DNC function iff every Π^0_1 class of positive measure admits an infinite X-computable homogeneous set. Such degrees can also be formulated in terms of strong immunity.

> **Definition 6.2.3.** Given a function $h : \mathbb{N} \to \mathbb{N}$, an infinite set A is *h-immune* if for every c.e. set W_e such that $W_e \subseteq A$, then $\operatorname{card} W_e \leq h(e)$. An infinite set is *effectively immune* if it is h-immune for some computable function $h : \mathbb{N} \to \mathbb{N}$. ◊

Theorem 6.2.4 (Jockusch [41])
Let X be a set. The following are equivalent.

1. *X computes a DNC function;*

2. X *computes an effectively immune set;*
3. X *computes a fixpoint-free function.*

Proof. $(1) \to (2)$: By Proposition 5.7.1, X computes a function $g : \mathbb{N}^2 \to \mathbb{N}$ such that for every $e, b \in \mathbb{N}$, if $\operatorname{card} W_e \leq b$, then $g(e,b) \notin W_e$. Let $D_0, D_1, \dots$ be a canonical enumeration of all non-empty finite sets. Let $h : \mathbb{N} \to \mathbb{N}$ be a partial computable function such that for every $e \in \mathbb{N}$, if $\operatorname{card} W_e > e$, then $D_{h(e)} \subseteq W_e$ and $\operatorname{card} D_{h(e)} = e + 1$. We shall construct an infinite increasing, X-computable sequence of integers $x_0 < x_1 < \dots$ such that for every $s \in \mathbb{N}$,

$$\forall e \leq s,\ (\operatorname{card} W_e > e \to D_{h(e)} \subsetneq \{x_i : i \leq s\}). \qquad (\star)$$

Then, $A = \{x_n : n \in \mathbb{N}\}$ is effectively immune, as witnessed by the identity function. Indeed, if $W_e \subseteq A$, then $\operatorname{card} W_e \leq e$. Assume $x_0 < \cdots < x_s$ is already constructed, satisfying $(\star)$. Let [8]

8: The left part $\{y : y \leq x_s\}$ of the union is to ensure that $x_{s+1} > x_s$, hence the set A is X-computable.

$$W_{v(s)} = \{y : y \leq x_s\} \cup \bigcup_{e \leq s+1 \,\wedge\, h(e)\downarrow} D_{h(e)}$$

Note that the function $v : \mathbb{N} \to \mathbb{N}$ is X-computable, and $\operatorname{card} W_{v(s)} \leq x_s + 1 + \sum_{n \leq s+2} n$, so, letting $x_{s+1} = g(v(s), x_s + 1 + \sum_{n \leq s+2} n)$, we have $x_{s+1} \notin W_{v(s)}$. In particular, $x_{s+1} > x_s$ and $x_0, \dots, x_{s+1}$ satisfies $(\star)$. This completes the construction.

$(2) \to (3)$: Let $A \leq_T X$ be an h-effectively immune set, for some computable function $h : \mathbb{N} \to \mathbb{N}$. Let $f : \mathbb{N} \to \mathbb{N}$ be an X-computable function such that $W_{f(e)}$ is the set of the $h(e) + 1$ first elements of A. We claim that f is a fixpoint-free function. Suppose for the contradiction that $W_{f(e)} = W_e$ for some $e \in \mathbb{N}$. Then $W_e \subseteq A$, but $\operatorname{card} W_e > h(e)$, contradiction.

$(3) \to (1)$: Let $f \leq_T X$ be a fixpoint-free function. Let $g : \mathbb{N} \to \mathbb{N}$ be the X-computable function such that for every n, $g(n)$ creates the code e_n of the function $m \mapsto \Phi_{\Phi_n(n)}(m)$[9], and outputs $f(e_n)$. We claim that g is DNC. Suppose for the contradiction that $g(n) = \Phi_n(n)$ for some $n \in \mathbb{N}$. Then by definition of g, $f(e_n) = \Phi_n(n)$. In particular, $\Phi_{f(e_n)} = \Phi_{\Phi_n(n)} = \Phi_{e_n}$. This contradicts the fact that f is fixpoint-free. ■

9: Here, $m \mapsto \Phi_{\Phi_n(n)}(m)$ is an abuse of notation for the program which, on input m, first executes $\Phi_n(n)$, and if it halts and outputs some e, executes $\Phi_e(m)$. In other words, the computation of $\Phi_n(n)$ is not part of the computation of g, hence g is total even if $\Phi_n(n)\uparrow$.

6.3 Hyperimmunity and WKL

Immunity and its variants form a unifying language to express custom invariant enabling to separate statements from Ramsey theory. The difficulty to separate to statements P and Q is to find a notion of immunity which is strong enough to be preserved by P, but weak enough not to be preserved by Q. This strengthening can often be obtained by studying the combinatorial features of the forcing question for P.

Let us consider the case of weak König's lemma, which captures the notion of compactness. Suppose one wants to prove that WKL preserves 1 immunity. This proof will fail, but one will exploit this failure to design a custom invariant. Fix an infinite immune set A, and let $\mathcal{P} \subseteq 2^{\mathbb{N}}$ be a non-empty Π^0_1 class. The natural notion of forcing to build members of Π^0_1 classes is Jockusch-Soare forcing $(\mathbb{P}, \leq)$, that is, the set of all infinite computable binary trees partially ordered by inclusion. Given a Turing index $e \in \mathbb{N}$, one wants to force the following requirement:

$\mathcal{R}_e$: W_e^G is not an infinite subset of A.

Recall that Jockusch-Soare forcing admits the following natural forcing question for Σ^0_1 formulas: Given a Σ^0_1-formula $\varphi(G)$, let $T \mathrel{?\vdash} \varphi(G)$ hold if there is some level $\ell \in \mathbb{N}$ such that for every $\sigma \in T \cap 2^\ell$, $\varphi(\sigma)$ holds. This forcing question is Σ^0_1-preserving and Σ^0_1-compact. The proof of $\mathcal{R}_e$ usually goes as follows: Given a condition $T \subseteq 2^{<\mathbb{N}}$ and a Turing index e, if T does not force W_e^G to be an infinite subset of A, then there is an extension $S \subseteq T$ forcing $\mathcal{R}_e$. If, on the other hand, T already forces W_e^G to be an infinite subset of A, then exploit the forcing question to compute an infinite subset of A, contradicting immunity of A.

Suppose we are in the second case. Given some $n \in \mathbb{N}$, one wants to find computably an element $x > n$ in A. The problem comes from the difference between the following two statements:

$$T \mathrel{?\vdash} \exists x(x > n \wedge x \in W_e^G) \quad \text{and} \quad \exists x\,(T \mathrel{?\vdash} x > n \wedge x \in W_e^G)$$

Assuming T forces W_e^G to be an infinite subset of A, the left statement holds, as otherwise, one would find an extension forcing W_e^G to be bounded by n, hence to be finite. On the other hand, the right statement does not hold in general. It might be that for each individual $x > n$, $T \mathrel{?\nvdash} x \in W_e^G$, but $T \mathrel{?\vdash}$ "W_e^G is infinite ". Thankfully, by Σ^0_1-compactness of the forcing question, one has the following implication

$$T \mathrel{?\vdash} \exists x(x > n \wedge x \in W_e^G) \quad \rightarrow \quad \exists F \text{ finite } (T \mathrel{?\vdash} \min F > n \wedge F \cap W_e^G \neq \emptyset)$$

Moreover, for any such F, we claim that $A \cap F \neq \emptyset$. Indeed, by definition of the forcing question, there is an extension $S \subseteq T$ forcing $F \cap W_e^G \neq \emptyset$, but S also forces $W_e^G \subseteq A$. Last, since the forcing question is Σ^0_1-preserving, for every n, one can computably find some F_n such that $F_n \cap A \neq \emptyset$ and $\min F_n > n$. In order to obtain a contradiction, one therefore must assume that no infinite subset of A can be approximated by finite sets, hence that A is hyperimmune. It happens that this is a sufficient invariant. Indeed, a finite union of finite sets is again a finite set.[10]

Statements from Ramsey theory do not usually imply weak König's lemma, and therefore might preserve a weaker form of immunity. For instance, the "compactness part" of Ramsey's theorem for pairs is the Ramsey-type weak König's lemma (RWKL).[11] However, it is often not necessary to consider the optimal invariant, and in many cases, on works with variants of hyperimmunity as soon as the statement contains some amount of compactness.

10: The computably dominated basis theorem for Π^0_1 classes is a much stronger form of preservation of 1 hyperimmunity, in the sense that every non-empty Π^0_1 class $\mathcal{P} \subseteq 2^{\mathbb{N}}$ has a member G such that every hyperimmune function is G-hyperimmune.

11: This sentence has to be taken in an informal sense. On one hand, $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \rightarrow \mathsf{RWKL}$, so the compactness part of RT^2_2 is at least RWKL. For the converse, the usual notion of forcing for Ramsey's theorem for pairs with a good first-jump control can be done with reservoirs restricted to any ω-model of $\mathsf{RCA}_0 + \mathsf{RWKL}$.

6.4 Erdős-Moser theorem

Let us step up and separate two statements from Ramsey's theory with very similar combinatorics: the Erdős-Moser theorem and Ramsey's theorem for pairs. The *Erdős-Moser theorem* is a statement about tournaments at the intersection of graph theory and Ramsey theory. A *tournament*[12] over an infinite domain $D \subseteq \mathbb{N}$ is an irreflexive binary relation $T \subseteq D^2$ such that for every $a, b \in D$ with $a \neq b$, $T(a, b)$ iff $\neg T(b, a)$. The tournament T is *transitive* if for all $a, b, c \in D$, if $T(a, b)$ and $T(b, c)$ hold, then $T(a, c)$ also holds.[13] A

12: This formalizes real-world tournaments: Intuitively, $T(a, b)$ if Player a beats Player b in a tournament. In general, a tournament is not transitive, that is, it might be that Player a beats Player b, who beats Player c, who himself beats Player a.

13: It is important to note that transitivity is a property over $[D]^3$. Thus, if a tournament is not transitive, then it is witnessed by a 3-tuple of elements of D.

sub-tournament of T is the restriction of T to a subdomain $D_1 \subseteq D$. Thus, given T, a sub-tournament is fully specified by the sub-domain D_1, and is therefore identified with it, and we say that D_1 is T*-transitive* if T is transitive on D_1.

The Erdős-Moser theorem states that every infinite tournament admits an infinite transitive subtournament. It can be seen as a Π^1_2 problem EM whose instances are tournaments on $\mathbb{N}$, and whose solutions are infinite domains on which the tournament is transitive. It follows from Ramsey's theorem for pairs and two colors by defining, given a tournament T on $\mathbb{N}$, a coloring $f : [\mathbb{N}]^2 \to 2$ such that for every $a < b$, $f(a, b) = 1$ iff $T(a, b)$. Then any infinite f-homogeneous set is T-transitive.[14]

14: The Erdős-Moser theorem was first studied in reverse mathematics by Bovykin and Weiermann [42]. Lerman, Solomon and Towsner [43] proved that EM is strictly weaker than RT^2_2 over RCA_0, later simplified by Patey [44].

Recall from Section 5.3 that RWKL is the Π^1_2 problem whose instances are infinite binary trees, and whose solutions are infinite homogeneous sets.[15] The following lemma shows that EM has the same amount of compactness as RT^2_2.

15: By Definition 5.3.1, given an infinite tree $T \subseteq 2^{<\mathbb{N}}$, a finite set $F \subseteq \mathbb{N}$ is T-*homogeneous* for color $i < 2$ if $\{\sigma \in T : (\forall x \in F)\sigma(x) = i\}$ is infinite. An infinite set H is T*-homogeneous* if every finite subset of H is T-homogeneous.

Exercise 6.4.1 (Bienvenu, Patey and Shafer [37]). Let $T \subseteq 2^{<\mathbb{N}}$ be an infinite binary tree. For each $s \in \mathbb{N}$, let σ_s be the left-most element of T of length s. Define a tournament T as follows: For $x < s$, if $\sigma_s(x) = 1$, then $R(x, s)$ holds and $R(s, x)$ fails. Otherwise, if $\sigma_s(x) = 0$, then $R(x, s)$ fails and $R(s, x)$ holds. Show that every infinite transitive subtournament computes an infinite T-homogeneous set. ★

Looking at the standard notion of forcing for Ramsey's theorem for pairs and for the Erdős-Moser theorem, the combinatorics are very similar, except that Ramsey's theorem for pairs is a disjunctive statement. Forcing multiple requirements is not an issue for the Erdős-Moser theorem. On the other hand, the situation for disjunctive statements is more delicate: if one forces requirements of the form $\mathcal{R} \vee \mathcal{R}$ and $\mathcal{S} \vee \mathcal{S}$, it might be that the $\mathcal{R}$-requirements and the $\mathcal{S}$-requirements are not satisfied on the same side.[16] This motivates the following definition:

16: It is sometimes possible to satisfy multiple requirements using a pairing argument, by forcing all the possible disjunctive pairs: $\mathcal{R} \vee \mathcal{R}$, $\mathcal{S} \vee \mathcal{S}$, $\mathcal{R} \vee \mathcal{S}$ and $\mathcal{S} \vee \mathcal{R}$.

Definition 6.4.2. A problem P admits *preservation of k hyperimmunities* if for every set Z and every k-tuple of Z-hyperimmune functions $f_0, \dots, f_{k-1}$, every Z-computable instance X of P admits a solution Y such that each f_i is $Z \oplus Y$-hyperimmune. ◊

We now prove that the Erdős-Moser theorem admits preservation of ω hyperimmunities.

Theorem 6.4.3 (Patey [44])
Let $h_0, h_1, \dots$ be a countable collection of hyperimmune functions, and let $T \subseteq \mathbb{N}^2$ be a computable tournament. There is an infinite T-transitive subtournament $G \subseteq T$ such that every h_i is G-hyperimmune.

Proof. Given two sets $E, F \subseteq \mathbb{N}$, we write $E \to_T F$ if for every $x \in E$ and every $y \in F$, $T(x, y)$. A set X is in a *minimal T-interval* of F if for every $a \in F$, either $\{a\} \to_T X$, or $X \to_T \{a\}$.[17]

17: One can actually define the notion of T-interval $(a, b)_T$ to be the set of all $x \in \mathbb{N}$ such that $T(a, x)$ and $T(x, b)$ (see [43]), but for our purpose, it is sufficient to work with a coarser definition.

Consider the notion of forcing whose *conditions*[18] are Mathias conditions (σ, X) such that

18: One would naturally be tempted to define a condition as a pair satisfying Items 1 and 3. Actually, Item 2 is already sufficient to ensure extendibility of the stem, but it requires some extra work. With the actual definition, one can simply apply the Erdős-Moser theorem to $T{\restriction}[X]^2$ to obtain an infinite T-transitive subset $Y \subseteq X$, and thanks to Item 1 and Item 2, $\sigma \cup Y$ is T-transitive.

1. $\sigma \cup \{x\}$ is T-transitive for every $x \in X$;
2. X is in a minimal T-interval of σ;[19]

19: Note that this property can be obtained for free by considering the map $g : X \to 2^{|\sigma|}$ which to x associates the string ρ of length $|\sigma|$ such that for every $y < |\sigma|$, $\rho(y) = 1$ iff $T(y, x)$ holds. By the pigeonhole principle, there is an infinite X-computable g-homogeneous subset $Y \subseteq X$. Any such Y is in a minimal T-interval of σ.

3. h_i is X-hyperimmune for every $i \in \mathbb{N}$.

The notion of extension is exactly Mathias extension. Every filter $\mathcal{F}$ induces a set $G_{\mathcal{F}}$ defined by $\bigcup\{\sigma : (\sigma, X) \in \mathcal{F}\}$. The following lemma shows that $G_{\mathcal{F}}$ is infinite for every sufficiently generic filter $G_{\mathcal{F}}$.

Lemma 6.4.4. Let $p = (\sigma, X)$ be a condition. There is an extension (τ, Y) of p and some $n > |\sigma|$ such that $n \in \tau$. ★

Proof. Pick any $n \in X$. Let $\tau = \sigma \cup \{n\}$, and Y be either $\{x \in X : T(n, x)\}$ or $\{x \in X : T(x, n)\}$, depending on which one is infinite. Then, $(\tau, Y \setminus \{0, \dots, n-1\})$ is an extension of p such that $n \in \tau$. ■

This notion of forcing admits a non-disjunctive, Σ^0_1-preserving, Σ^0_1-compact forcing question.

Definition 6.4.5. Let $p = (\sigma, X)$ be a condition, and let $\varphi(G)$ be a Σ^0_1-formula. Let $p \mathrel{?\vdash} \varphi(G)$ hold if for every 2-partition $Z_0 \sqcup Z_1 = X$, there is some $i < 2$ and some finite T-transitive set $\rho \subseteq Z_i$ such that $\varphi(\sigma \cup \rho)$ holds.[20] ◇

20: Note the similarity of this forcing question with the one from Exercise 3.4.12.

Note that by compactness, the forcing question is $\Sigma^0_1(X)$. The following lemma states that the forcing question meets its specification.

Lemma 6.4.6. Let $p = (\sigma, X)$ be a condition, and let $\varphi(G)$ be a Σ^0_1-formula.

1. If $p \mathrel{?\vdash} \varphi(G)$, then there is an extension $q \leq p$ forcing $\varphi(G)$;
2. If $p \mathrel{?\nvdash} \varphi(G)$, then there is an extension $q \leq p$ forcing $\neg\varphi(G)$. ★

Proof. Suppose first $p \mathrel{?\vdash} \varphi(G)$. Then, by compactness, there is some threshold $\ell \in \mathbb{N}$ such that for every 2-partition $Z_0 \sqcup Z_1 = X \upharpoonright \ell$, there is some $i < 2$ and some finite T-transitive set $\rho \subseteq Z_i$ such that $\varphi(\sigma \cup \rho)$ holds. For every $x \in X \setminus \{0, \dots, \ell\}$, let σ_x be the binary string of length ℓ such that for every $y < \ell$, $T(y, x) = \sigma_x(y)$. By the pigeonhole principle, there is some string σ of length ℓ and an infinite X-computable subset $Y \subseteq X \setminus \{0, \dots, \ell\}$ such that for $\sigma = \sigma_x$ for every $x \in Y$. Let $Z_i = X \cap \{y : \sigma(y) = i\}$ for each $i < 2$. By assumption, there is some $i < 2$ and some finite T-transitive set $\rho \subseteq Z_i$ such that $\varphi(\sigma \cup \rho)$ holds. We claim that $(\sigma \cup \rho, Y)$ is an extension of p forcing $\varphi(G)$.

Suppose now $p \mathrel{?\nvdash} \varphi(G)$. Let $\mathscr{C}$ be the $\Pi^0_1(X)$ class of all $Z_0 \oplus Z_1$ such that, $Z_0 \sqcup Z_1 = X$ and for every $i < 2$ and every finite T-transitive set $\rho \subseteq Z_i$, $\varphi(\sigma \cup \rho)$ does not hold. By the computably dominated basis theorem (see Jockusch and Soare [9]), there is some 2-partition $Z_0 \sqcup Z_1 = X$ such that $Z_0 \oplus Z_1 \oplus X$ is computably X-dominated. In particular, each h_i is $Z_0 \oplus Z_1 \oplus X$-hyperimmune. Let $i < 2$ be such that Z_i is infinite. Then (σ, Z_i) is an extension of p forcing $\neg\varphi(G)$. ■

The following lemma is an adaptation of Theorem 3.6.4.

Lemma 6.4.7. Let $p = (\sigma, X)$ be a condition. For every Turing index e and every $i \in \mathbb{N}$, there is an extension $q \leq p$ forcing Φ^G_e not to dominate h_i.[21] ★

21: By this, we mean forcing either Φ^G_e to be partial, or $\Phi^G_e(x) < h_i(x)$ for some $x \in \mathbb{N}$.

Proof. Let $?\vdash$ be the forcing question of Definition 6.4.5. Suppose first that $p \mathrel{?\nvdash} \exists v \Phi_e^G(x)\downarrow = v$ for some $x \in \mathbb{N}$. Then by Lemma 6.4.6(2), there is an extension $q \leq p$ forcing $\Phi_e^G(x)\uparrow$, and we are done. Suppose now that for every $x \in \mathbb{N}$, $p \mathrel{?\vdash} \exists v \Phi_e^G(x)\downarrow = v$. By Σ^0_1-compactness of the forcing question, for every $x \in \mathbb{N}$, there is a finite set $F_x \subseteq \mathbb{N}$ such that $p \mathrel{?\vdash} \exists v \in F_x\ \Phi_e^G(x)\downarrow = v$. Let $g : \mathbb{N} \to \mathbb{N}$ be the function which on input x, looks for some finite set F_x such that $p \mathrel{?\vdash} \exists v \in F_x\ \Phi_e^G(x)\downarrow = v$ and outputs $\max F_x$. Such a function is total by hypothesis, and X-computable since the forcing question is $\Sigma^0_1(X)$. Since h_i is X-hyperimmune, $g(x) < h_i(x)$ for some $x \in \mathbb{N}$. By Lemma 6.4.6(1), there is an extension $q \leq p$ forcing $\exists v \in F_x \Phi_e^G(x)\downarrow = v$. Since $h_i(x) > \max F_x$, q forces $\Phi_e^G(x)\downarrow < h_i(x)$. ■

We are now ready to prove Theorem 6.4.3. Let $\mathcal{F}$ be a sufficiently generic filter for this notion of forcing,. By Lemma 6.4.4, $G_{\mathcal{F}}$ is infinite. Moreover, by Lemma 6.4.7, h_i is $G_{\mathcal{F}}$-hyperimmune for every $i \in \mathbb{N}$. This completes the proof of Theorem 6.4.3. ■

The following proposition shows that RT^2_2 does not admit preservation of 2 hyperimmunities.

Proposition 6.4.8. There exists two hyperimmune functions $g_0, g_1 : \mathbb{N} \to \mathbb{N}$ and a computable coloring $f : [\mathbb{N}]^2 \to 2$ such that for every infinite f-homogeneous set H for color i, g_i is not H-hyperimmune. ⋆

Proof. Let $A_0 \sqcup A_1$ be a Δ^0_2 2-partition such that A_0 and A_1 are hyperimmune, and let $g_i = p_{A_i}$ be the principal function of A_i for each $i < 2$. By Shoenfield's limit lemma, there is a computable function $f : [\mathbb{N}]^2 \to 2$ such that for every x, $\lim_y f(x, y)$ exists, and equals i iff $x \in A_i$. For every infinite f-homogeneous set H for color i, $H \subseteq A_i$. In particular, p_H dominates g_i, so g_i is not H-hyperimmune. ■

Corollary 6.4.9 (Lerman, Solomon and Towsner [43])
EM *does not imply* RT^2_2 *over* RCA_0.

Proof. Immediate by Proposition 6.4.8, Theorem 6.4.3 and Corollary 6.1.4.■

Consider three kinds of requirement $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$. Suppose one can construct solutions to Ramsey's theorem for pairs and two colors by satisfying requirements of type $\mathcal{R} \vee \mathcal{R}$, $\mathcal{S} \vee \mathcal{S}$ and $\mathcal{T} \vee \mathcal{T}$. By the pigeonhole principle, there must be a side preserving two kinds of requirements simultaneously. In the case of preservation of hyperimmunities, it yields that, given 3 hyperimmune functions, one can always construct solutions to computable instances of RT^2_2 while preserving two among the three hyperimmunities simultaneously. We leave the proofs as an exercise.

Exercise 6.4.10 (Patey [45]). A problem P admits *preservation of ℓ among k hyperimmunities* if for every set Z and every k-tuple of Z-hyperimmune functions $f_0, \dots, f_{k-1}$, every Z-computable instance X of P admits a solution Y and some finite set $F \in [k]^\ell$ such that for each $i \in F$, f_i is $Z \oplus Y$-hyperimmune.

1. Show that RT^2_3 does not admit preservation of 3 among 3 hyperimmuni-

ties.[22]

2. Show that RT^2_2 admits preservation of 2 among 3 hyperimmunities.[23] ⋆

22: Hint: Adapt the proof of Proposition 6.4.8).

23: Hint: Adapt the proof of Theorem 6.4.3, but with the notion of forcing of Exercise 3.4.12.

6.5 Partial orders

Partial orders also provide a good family of Ramsey-type theorems requiring custom preservations properties. A *partial order* is a pair $\mathscr{P} = (D, <_{\mathscr{P}})$, where $D \subseteq \mathbb{N}$ and $<_{\mathscr{L}}$ is an irreflexive transitive binary relation over D. A set $X \subseteq D$ is an *chain (antichain)* if every two elements of X are comparable (incomparable) over $<_{\mathscr{P}}$. A set $X \subseteq D$ is an *ascending (descending) sequence* if for every $x, y \in X$, $x < y$ iff $x <_{\mathscr{P}} y$ ($x >_{\mathscr{P}} y$). The *Chain AntiChain* principle[24] (CAC) is the Π^1_2-problem whose instances are partial orders over $\mathbb{N}$ and whose solutions are infinite chains or infinite antichains.

24: This principle was studied by Herrmann [21] and Hirschfeldt and Shore [23] in reverse mathematics.

Exercise 6.5.1 (Hirschfeldt and Shore [23]). Show that RCA_0+CAC proves that every partial order on $\mathbb{N}$ admits either an infinite ascending or descending sequence, or an infinite antichain. ⋆

Exercise 6.5.2 (Hirschfeldt and Shore [23]). A coloring $f : [\mathbb{N}]^2 \to k$ is *transitive for color* $i < k$ if for every $x < y < z$ such that $f(x, y) = f(y, z) = i$, then $f(x, z) = i$. Show that CAC is equivalent over RCA_0 to the statement "For every transitive coloring $f : [\mathbb{N}]^2 \to 2$ for some color, there is an infinite f-homogeneous set." ⋆

Exercise 6.5.3 (Herrmann [21]). Construct a computable partial order on $\mathbb{N}$ with no infinite computable chain or antichain. ⋆

As it happens, building either an ascending or a descending sequence has better combinatorial properties than building a chain. We shall therefore build a strong solution to CAC, in the sense of Exercise 6.5.1. The corresponding notion of forcing admits a forcing question for Σ^0_1 formulas which is strongly Σ^0_1-compact, in that if $p \mathrel{?{\vdash}} \exists x \varphi(G, x)$, then there is a set F of size 3 such that $p \mathrel{?{\vdash}} (\exists x \in F)\varphi(G, x)$. Following the process of Section 6.3, this yields the following notion of immunity:

> **Definition 6.5.4.** A *c.e. k-array* is a c.e. array $\{D_{f(n)} : n \in \mathbb{N}\}$ such that $\operatorname{card} D_{f(n)} \leq k$ for each n. An infinite set $A \subseteq \mathbb{N}$ is *k-immune* if for every c.e. k-array $\{D_{f(n)} : n \in \mathbb{N}\}$, there is some n such that $A \cap D_{f(n)} = \emptyset$. A set A is *constant-bound immune (c.b-immune)* if it is k-immune for every $k \in \mathbb{N}$. ◇

Constant-bound immunity is a strong form of immunity. The following exercise shows that two notions coincide on co-c.e. sets.

Exercise 6.5.5. Let A be a co-c.e. set. Show that A is immune iff A is c.b-immune. ⋆

As usual, every notion of immunity induces a preservation property.

> **Definition 6.5.6.** A problem P admits *preservation of 1 c.b-immuniy* if for every set Z and every c.b-Z-immune set A, every Z-computable instance X of P admits a solution Y such that A is c.b-$Z \oplus Y$-immune. ◇

We now prove that CAC admits preservation of 1 c.b-immuniy.

Theorem 6.5.7 (Patey [46])
Let A be a c.b-immune set, and $\mathcal{P} = (\mathbb{N}, <_{\mathcal{P}})$ be a computable partial order. Then there is either an infinite ascending or descending sequence G, or an infinite antichain G such that A is c.b-G-immune.

Proof. Consider the notion of forcing whose *conditions* are 4-tuples $(\sigma_0, \sigma_1, \sigma_2, X)$, where

1. (σ_i, X) is a Mathias condition for each $i < 3$;
2. $\sigma_0 \cup \{x\}$, $\sigma_1 \cup \{x\}$ and $\sigma_2 \cup \{x\}$ form respectively an ascending sequence, a descending sequence and an antichain, for each $x \in X$;
3. X is computable.[25]

25: Having a notion of forcing with a good first-jump control while keeping the reservoir computable is a good indicator that the statement does not imply any form of compactness.

A condition $(\tau_0, \tau_1, \tau_2, Y)$ *extends* $(\sigma_0, \sigma_1, \sigma_2, X)$ if (τ_i, Y) Mathias extends (σ_i, X) for every $i < 3$. One can therefore see a condition as three simultaneous Mathias conditions sharing a same reservoir. Every filter $\mathcal{F}$ induces three sets: $G_{0,\mathcal{F}}$, $G_{1,\mathcal{F}}$ and $G_{2,\mathcal{F}}$, defined by $G_{i,\mathcal{F}} = \bigcup\{\sigma_i : (\sigma_0, \sigma_1, \sigma_2, X) \in \mathcal{F}\}$.

As in the proof of Theorem 3.4.6, if $\mathcal{F}$ is a sufficiently generic filter, then $G_{i,\mathcal{F}}$ is not necessarily infinite. We shall therefore make the following hypothesis:

> (H1): For every infinite computable set X, there is some $x_0, x_1, x_2 \in X$ such that $\{y \in X : x_0 <_{\mathcal{P}} y\}$, $\{y \in X : x_1 >_{\mathcal{P}} y\}$ and $\{y \in X : x_2 \mid_{\mathcal{P}} y\}$ are all infinite.

If the (H1) hypothesis fails for some set X, one can computably thin it out to obtain an infinite subset $Y \subseteq X$ which avoids one of the three behaviors. One then restarts the construction with conditions whose reservoirs are subsets of Y. The conditions will then have less stems, and the forcing questions must be adapted accordingly.

Lemma 6.5.8. Suppose (H1) holds. Let $p = (\sigma_0, \sigma_1, \sigma_2, X)$ be a condition and $i < 3$. There is an extension $(\tau_0, \tau_1, \tau_2, Y)$ of p and some $x > |\sigma_i|$ such that $x \in \tau_i$. ⋆

Proof. Say $i = 0$. Then two other cases are similar. By (H1), there is some $x_0 \in X$ such that $Y = \{y \in X : x_0 <_{\mathcal{P}} y\}$ is infinite. Let $\tau_0 = \sigma_0 \cup \{x_0\}$, and $\tau_i = \sigma_i$ otherwise. Then, $(\tau_0, \tau_1, \tau_2, Y)$ is an extension of p such that $x_0 \in \tau_0$. ■

We now define a disjunctive forcing question for Σ^0_1-formulas. Given a condition $p = (\sigma_0, \sigma_1, \sigma_2, X)$, a *split triple* is a 3-tuple (ρ_0, ρ_1, ρ_2) such that $\rho_i \subseteq X$ for each $i < 3$, ρ_0 is ascending, ρ_1 is descending, ρ_2 is an antichain, and for every $x \in \rho_2$, $\max_{\mathcal{P}}(\rho_0) <_{\mathcal{P}} x <_{\mathcal{P}} \min_{\mathcal{P}}(\rho_1)$.[26]

26: In other words, every element of the ascending sequence ρ_0 is below (with respect to $<_{\mathcal{P}}$) every element of the antichain ρ_2, and every element of ρ_2 is below every element of the descending sequence ρ_1.

> **Definition 6.5.9.** Let $p = (\sigma_0, \sigma_1, \sigma_2, X)$ be a condition and $\varphi_0(G)$, $\varphi_1(G)$ and $\varphi_2(G)$ be three Σ^0_1-formulas. Let $p \operatorname{?\vdash} \varphi_0(G_0) \vee \varphi_1(G_1) \vee \varphi_2(G_2)$ hold if there is a split triple (ρ_0, ρ_1, ρ_2) such that for each $i < 3$, $\varphi_i(\sigma_i \cup \rho_i)$ holds. ◇

Note that being a split triple is a decidable predicate, hence the forcing question is Σ^0_1-preserving. The following lemma shows that the forcing question meets its specification.

Lemma 6.5.10. Let $p = (\sigma_0, \sigma_1, \sigma_2, X)$ be a condition and $\varphi_0(G)$, $\varphi_1(G)$ and $\varphi_2(G)$ be three Σ^0_1-formulas.

1. If $p \operatorname{?\vdash} \varphi_0(G_0) \vee \varphi_1(G_1) \vee \varphi_2(G_2)$, then there is some $i < 3$ and some

extension $q \leq p$ forcing $\varphi_i(G_i)$.
2. If $p \mathrel{?\nvdash} \varphi_0(G_0) \vee \varphi_1(G_1) \vee \varphi_2(G_2)$, then there is some $i < 3$ and some extension $q \leq p$ forcing $\neg\varphi_i(G_i)$. ⋆

Proof. Suppose first $p \mathrel{?\vdash} \varphi_0(G_0) \vee \varphi_1(G_1) \vee \varphi_2(G_2)$ holds, as witnessed by some split triple (ρ_0, ρ_1, ρ_2). By the pigeonhole principle, there is some infinite X-computable subset $Y \subseteq X$ such that for every $x \in \rho_0 \cup \rho_1 \cup \rho_2$, either for every $y \in Y$, $x <_{\mathcal{P}} y$, or for every $y \in Y$, $x >_{\mathcal{P}} y$, or for every $y \in Y$, $x|_{\mathcal{P}} y$. We say that x is *small* if it is on the first case, *large* if it is on the second case, and *isolated* if it is on the third case. If every $x \in \rho_2$ is isolated, then the condition $(\sigma_0, \sigma_1, \sigma_2 \cup \rho_2, Y)$ is an extension of p forcing $\varphi_2(G_2)$. If some $x \in \rho_2$ is small, then every element in ρ_0 is small, so $(\sigma_0 \cup \rho_0, \sigma_1, \sigma_2, Y)$ is an extension of p forcing $\varphi_0(G_0)$. Last, if some $x \in \rho_2$ is large, then every element in ρ_1 is large, thus $(\sigma_0, \sigma_1 \cup \rho_1, \sigma_2, Y)$ is an extension of p forcing $\varphi_1(G_1)$.

Suppose now $p \mathrel{?\nvdash} \varphi_0(G_0) \vee \varphi_1(G_1) \vee \varphi_2(G_2)$. We have two cases. Case 1: there are two sets $\rho_0, \rho_1 \subseteq X$ such that ρ_0 is ascending, ρ_1 is descending, and the set $Y = \{x \in X : \max_{\mathcal{P}} \rho_0 <_{\mathcal{P}} x <_{\mathcal{P}} \min_{\mathcal{P}} \rho_1\}$ is infinite. Then the condition $q = (\sigma_0 \cup \rho_0, \sigma_1 \cup \rho_1, \sigma_2, Y)$ is an extension forcing $\neg\varphi_2(G_2)$. Indeed, if there is an extension $r = (\tau_0, \tau_1, \tau_2, Z)$ of q such that $\varphi_2(\tau_2)$ holds, then, letting $\rho_2 = \tau_2 \setminus \sigma_2$, the tuple (ρ_0, ρ_1, ρ_2) forms a split triple contradicting our hypothesis. Case 2: there are no such two sets. Then we claim that p already forces $\neg\varphi(G_0) \vee \neg\varphi(G_1)$. Indeed, if there is some extension $q = (\tau_0, \tau_1, \tau_2, Y)$ of p such that $\varphi_0(\tau_0)$ and $\varphi_1(\tau_1)$ both hold, then, letting $\rho_i = \tau_i \setminus \sigma_i$, the sets ρ_0, ρ_1 witness Case 1. Thus there is an extension of p forcing either $\neg\varphi(G_0)$, or $\neg\varphi(G_1)$. ■

By definition of the forcing question, if

$$p \mathrel{?\vdash} \exists x \varphi_0(G_0, x) \vee \exists x \varphi_1(G_1, x) \vee \exists x \varphi_2(G_2, x)$$

then there are three elements $n_0, n_1, n_2 \in \mathbb{N}$ such that

$$p \mathrel{?\vdash} \varphi_0(G_0, n_0) \vee \varphi_1(G_1, n_1) \vee \varphi_2(G_2, n_2)$$

This can be seen as some strong form of Σ^0_1-compactness, where the finite set is of size at most 3.

Lemma 6.5.11. Let $p = (\sigma_0, \sigma_1, \sigma_2, X)$ be a condition and $\Phi_{e_0}, \Phi_{e_1}, \Phi_{e_2}$ be three c.e. k-array functionals.[27] There is an extension q of p forcing $\Phi_{e_i}^{G_i}$ to be partial, or $\Phi_{e_i}^{G_i}(n)\downarrow \cap A = \emptyset$ for some $n \in \mathbb{N}$. ⋆

27: By this, we mean that for every oracle Z, if $\Phi_{e_i}^Z(n)\downarrow$, then its output is a finite set F of size at most k with $\min F > n$.

Proof. Suppose first that $p \mathrel{?\nvdash} \Phi_{e_0}^{G_0}(n)\downarrow \vee \Phi_{e_1}^{G_1}(n)\downarrow \vee \Phi_{e_2}^{G_2}(n)\downarrow$ for some n. Then by Lemma 6.5.10(2), there is an extension q of p forcing $\Phi_{e_i}^{G_i}(n)\uparrow$ for some $i < 3$.

Suppose now that for every $n \in \mathbb{N}$, $p \mathrel{?\vdash} \Phi_{e_0}^{G_0}(n)\downarrow \vee \Phi_{e_1}^{G_1}(n)\downarrow \vee \Phi_{e_2}^{G_2}(n)\downarrow$. Then for each $n \in \mathbb{N}$, there is some finite set E_n of size at most $3k$ such $p \mathrel{?\vdash} \Phi_{e_0}^{G_0}(n)\downarrow \subseteq E_n \vee \Phi_{e_1}^{G_1}(n)\downarrow \subseteq E_n \vee \Phi_{e_2}^{G_2}(n)\downarrow \subseteq E_n$. Moreover, since the forcing question is Σ^0_1-preserving, then the map $n \mapsto E_n$ is computable, so $(E_n : n \in \mathbb{N})$ forms a c.e. $3k$-array. By c.b-immunity of A, there is some $n \in \mathbb{N}$ such that $E_n \cap A = \emptyset$. By Lemma 6.5.10(1), there is an extension q of p forcing $\Phi_{e_i}^{G_i}(n)\downarrow \subseteq E_n$ for some $i < 3$. In particular, q forces $\Phi_{e_i}^{G_i}(n)\downarrow \cap A = \emptyset$. ■

We are now ready to prove Theorem 6.5.7 in the case (H1) holds. Let $\mathcal{F}$ be a sufficiently generic filter for this notion of forcing. For each $i < 3$, let $G_i = G_{\mathcal{F},i}$. By Lemma 6.5.8, G_i is infinite for every $i < 3$. By Lemma 6.5.11, there is some $i < 3$ such that A is c.b-G_i-immune. The case where (H1) does not hold is left to the reader, and consists in a degenerate forcing construction. This completes the proof of Theorem 6.5.7. ■

Looking at the proof of Theorem 6.5.7, the core of the combinatorics lies in the existence of a Σ^0_1-preserving forcing question which admits the following strong form of Σ^0_1-compactness.

Definition 6.5.12. Given a notion of forcing $(\mathbb{P}, \leq)$, a forcing question is *constant-bound Σ^0_n-compact* if for every $p \in \mathbb{P}$, there is some $k \in \mathbb{N}$ such that for every Σ^0_n formula $\varphi(G, x)$, if $p \mathrel{?\vdash} \exists x \varphi(G, x)$ holds, then there is a finite set $F \subseteq \mathbb{N}$ of size k such that $p \mathrel{?\vdash} \exists x \in F\ \varphi(G, x)$. ◊

We leave the following abstract theorem of preservation of 1 c.b-immunity as an exercise.

Exercise 6.5.13. Let $(\mathbb{P}, \leq)$ be a notion of forcing with a constant-bound Σ^0_1-compact, Σ^0_1-preserving forcing question. Show that for every c.b-immune set A and every sufficiently generic filter $\mathcal{F}$, A is c.b-immmune relative to $G_{\mathcal{F}}$.★

Let DNC be the Π^1_2-problem whose instances are any sets, and, given a set X, a solution is a DNC function relative to X. Recall that by Section 5.7, DNC can be seen as a form of compactness statement, in that it is equivalent to the Ramsey-type weak weak König's lemma (see Proposition 5.7.2). The following theorem therefore shows, as expected, that DNC not to admit preservation of constant-bound immunity.

Theorem 6.5.14 (Patey [46])
There is a Δ^0_2, c.b-immune set $A \subseteq \mathbb{N}$ such that every DNC function computes an infinite subset.

Proof. Let $\mu_{\emptyset'}$ be the modulus of $\emptyset'$, that is, such that $\mu_{\emptyset'}(x)$ is the minimum stage s at which $\emptyset'_s \upharpoonright x = \emptyset' \upharpoonright x$.[28]

Computably split $\mathbb{N}$ into countably many columns $X_0, X_1, \dots$ of infinite size. For example, set $X_i = \{\langle i, n\rangle : n \in \mathbb{N}\}$ where $\langle \cdot, \cdot \rangle$ is the Cantor bijection from $\mathbb{N}^2$ to $\mathbb{N}$. For each i, let F_i be the set of the $\mu_{\emptyset'}(i)$ first elements of X_i. The sequence $F_0, F_1, \dots$ is $\emptyset'$-computable. Assume for now that we have defined a c.e. set W such that the Δ^0_2 set $A = \bigcup_i F_i \setminus W$ is c.b-immune, and such that $|X_i \cap W| \leq i$. We claim that every DNC function computes an infinite subset of A.

Let f be any DNC function. By Proposition 5.7.1, f computes a function $g(\cdot,\cdot,\cdot)$ such that whenever $|W_e| \leq n$, then $g(e, n, i) \in X_i \setminus W_e$.[29] For each i, let e_i be the index of the c.e. set $W_{e_i} = W \cap X_i$, and let $n_i = g(e_i, i, i)$. Since $|X_i \cap W| \leq i$, then $|W_{e_i}| \leq i$, so $n_i = g(e_i, i, i) \in X_i \setminus W_{e_i}$, which implies $n_i \in X_i \setminus W$. We then have two cases.

- Case 1: $n_i \in F_i$ for infinitely many i's. One can f-computably find infinitely many of them since $\mu_{\emptyset'}$ is left-c.e. and the sequence of the n's is f-computable. Therefore, one can f-computably find an infinite subset of $\bigcup_i F_i \setminus W = A$.

28: Note that this modulus is *left-c.e.*, that is, there is a uniformly computable sequence of functions $g_0, g_1, \dots$ such that for every $s, x \in \mathbb{N}$, $g_s(x) \leq g_{s+1}(x) \leq \mu_{\emptyset'}(x)$. In other words, the set $\{(x, y) : y < \mu_{\emptyset'}(x)\}$ is c.e.

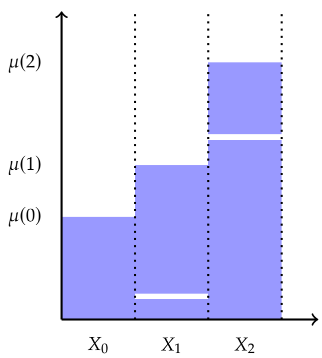

Figure 6.1: The set A (in blue) is a countable union of some finite initial segments $F_0, F_1, \dots$ of the columns $X_0, X_1, \dots$, from which finitely many elements have been removed in a c.e. way. The holes in the columns are the elements of W.

29: The function g can be obtained from Proposition 5.7.1 by "renaming" the elements of X_i using the bijection between X_i and $\mathbb{N}$.

- Case 2: $n_i \in F_i$ for only finitely many i's. Then the sequence of the n_i's eventually dominates the modulus function $\mu_{\emptyset'}$, and therefore computes the halting set. Since the set A is Δ^0_2, f computes an infinite subset of A.

We now detail the construction of the c.e. set W. In what follows, interpret Φ_e as a partial computable sequence of finite sets such that if $\Phi_e(x)$ halts, then $\min(\Phi_e(x)) > x$. We need to satisfy the following requirements for each $e, k \in \mathbb{N}$:

$$\mathcal{R}_{e,k}: \quad \begin{array}{l} [\Phi_e \text{ total } \wedge (\forall i)(\forall^\infty x)(\Phi_e(x) \cap X_i = \emptyset)] \\ \rightarrow (\exists x)\,[|\Phi_e(x)| > k \vee \Phi_e(x) \subseteq W] \end{array}$$

We furthermore want to ensure that $|X_i \cap W| \leq i$ for each i. We can prove by induction over k that if $\mathcal{R}_{e,\ell}$ is satisfied for each $\ell \leq k$, then the set $A = \bigcup_i F_i \setminus W$ is k-immune. The case $k = 1$ is trivial, since if Φ_e is a total c.e. 1-array and $\exists^\infty x \Phi_e(x) \cap X_i \neq \emptyset$, then $\exists^\infty x \Phi_e(x) \subseteq X_i$, so $\exists x \Phi_e(x) \subseteq (X_i \setminus F_i) \subseteq \overline{A}$. For the case $k \geq 2$, assume that Φ_e is a total c.e. k-array. If the right-hand side of the implication $\mathcal{R}_{e,k}$ holds, then we are done, so suppose it does not hold. In particular, the set $Y_i = \{x : \Phi_e(x) \cap X_i \neq \emptyset\}$ is infinite for some $i \in \mathbb{N}$. Let $Z_i \subseteq Y_i$ be a computable infinite subset such that $\min Z_i > \max F_i$. Say $Z_i = \{x_0 < x_1 < \dots\}$. Since $x < \min(\Phi_e(x))$, then for every $n \in \mathbb{N}$, $F_i < \Phi_e(x_n)$, hence $\Phi_e(x_n) \cap X_i \subseteq \overline{A}$. Let $E_0 < E_1 < \dots$ be defined by $E_n = \Phi_e(x_n) \setminus X_i$. Then $|E_n| < k$ for every n, so by induction hypothesis, there is some n such that $E_n \cap A = \emptyset$. In particular, $\Phi_e(x_n) \cap A = \emptyset$.

We now explain how to satisfy $\mathcal{R}_{e,k}$ for each $e, k \in \mathbb{N}$. For each pair of indices $e, k \in \mathbb{N}$, let $i_{e,k} = \sum_{\langle e',k'\rangle \leq \langle e,k\rangle} k'$. A strategy for $\mathcal{R}_{e,k}$ *requires attention* at stage $s > \langle e, k\rangle$ if there is an $x < s$ such that $\Phi_{e,s}(x)\downarrow$, $|\Phi_{e,s}(x)| \leq k$, and $\Phi_{e,s}(x) \subseteq \bigcup_{j \geq i_{e,k}} X_j$. Then, the strategy enumerates all the elements of $\Phi_{e,s}$ in W, and is declared *satisfied*, and will never require attention again. First, notice that if Φ_e is total, outputs k-sets, and meets finitely many times each X_i, then it will require attention at some stage s and will be declared satisfied. Therefore each requirement $\mathcal{R}_{e,k}$ is satisfied. Second, suppose for the sake of contradiction that $|X_i \cap W| > i$ for some i. Let s be the stage at which it happens, and let $\langle e, k\rangle < s$ be the maximal pair such that $\mathcal{R}_{e,k}$ has enumerated some element of X_i in W. In particular, $i_{e,k} \leq i$. Since the strategy for $\mathcal{R}_{e',k'}$ enumerates at most k' elements in W,

$$\sum_{\langle e',k'\rangle \leq \langle e,k\rangle} k' \geq |X_i \cap W| > i \geq i_{e,k} = \sum_{\langle e',k'\rangle \leq \langle e,k\rangle} k'$$

Contradiction. ■

Corollary 6.5.15 (Hirschfeldt and Shore [23])
CAC *implies neither* DNC *nor* RT^2_2 *over* RCA_0.[30]

30: Actually, this separation was originally proven using DNC avoidance. However, the design c.b-immunity is more straightforward from an analysis for the combinatorial properties of the forcing question for CAC.

Proof. By Theorem 6.5.7, Theorem 6.5.14 and Corollary 6.1.4, CAC does not imply DNC over RCA_0. By Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp, and Slaman [47], $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \rightarrow$ DNC, so CAC does not imply RT^2_2 over RCA_0. ■

6.6 Linear orders

A *linear order* is a pair $\mathcal{L} = (D, <_{\mathcal{L}})$ where $D \subseteq \mathbb{N}$ and $<_{\mathcal{L}}$ is an irreflexive and transitive total binary relation over D. A set $X \subseteq D$ is an *ascending (descending) sequence* if for every $x, y \in X$, $x < y$ iff $x <_{\mathcal{L}} y$ ($x >_{\mathcal{L}} y$). Let ADS be the Π^1_2 problem whose instances are infinite linear orders over $\mathbb{N}$ and whose solutions are infinite ascending or descending sequences.

Exercise 6.6.1 (Hirschfeldt and Shore [23]). Show that $\mathsf{RCA}_0 \vdash \mathsf{CAC} \to \mathsf{ADS}$. ★

Exercise 6.6.2 (Hirschfeldt and Shore [23]). Let $\vec{R} = R_0, R_1, \dots$ be a countable sequence of sets. Let $\mathcal{L} = (\mathbb{N}, <_{\mathcal{L}})$ be the linear order defined by setting $x <_{\mathcal{L}} y$ iff $\langle R_i(x) : i \leq x\rangle <_{\mathtt{lex}} \langle R_i(y) : i \leq y\rangle$, where $<_{\mathtt{lex}}$ is the lexicographic order on $2^{<\mathbb{N}}$. Show that every infinite ascending or descending sequence of $\mathcal{L}$ is $\vec{R}$-cohesive. ★

The Ascending Descending Sequence plays a dual role with the Erdős-Moser theorem with respect to RT^2_2 in the following sense: Any coloring $f : [\mathbb{N}]^2 \to 2$ can be interpreted as a tournament $T \subseteq \mathbb{N}^2$ by letting $T(x, y)$ hold if $x < y$ and $f(\{x, y\}) = 1$, or if $x > y$ and $f(\{y, x\}) = 0$. Every infinite T-transitive sub-tournament $U \subseteq \mathbb{N}$ induces a linear order $(U, <_{\mathcal{U}})$ defined by $x <_{\mathcal{U}} y$ iff $T(x, y)$ holds. Then, every infinite ascending and descending sequence is f-homogeneous for colors 1 and 0, respectively.

Exercise 6.6.3 (Montálban, see [42]). Show that $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \leftrightarrow \mathsf{EM} \wedge \mathsf{ADS}$. ★

One can naturally ask whether a reversal exists, that is, whether ADS implies CAC over RCA_0. The goal of this section is to separate the two statements. The natural notion of forcing for ADS is a degenerate version of the notion of forcing for CAC used in Theorem 6.5.7. The combinatorics are therefore very similar, with one notable exception:

Definition 6.6.4. Given a notion of forcing $(\mathbb{P}, \leq)$ and a family of formulas Γ, a forcing question is Γ-*extremal* if for every formula $\varphi \in \Gamma$ and every condition $p \in \mathbb{P}$, if $p \mathrel{?\vdash} \varphi(G)$ then p forces $\varphi(G)$. ◊

By extension, we say that a forcing question for Σ^0_n-formulas is Π^0_n-*extremal* if for every Σ^0_n-formula φ and every condition $p \in \mathbb{P}$, if $p \mathrel{?\nvdash} \varphi(G)$, then p forces $\neg\varphi(G)$.

Contrary to CAC, the notion of forcing for ADS admits a disjunctive forcing question which satisfies some form of Π^0_1-extremality. This extremality can be exploited to force countably many Π^0_1 facts simultaneously, yielding the following notion of immunity.

Definition 6.6.5. A formula $\varphi(U, V)$ is *essential*[31] if for every $x \in \mathbb{N}$, there is a finite set $R > x$ such that for every $y \in \mathbb{N}$, there is a finite set $S > y$ such that $\varphi(R, S)$ holds. A pair of sets $A_0, A_1 \subseteq \mathbb{N}$ is *dependently X-hyperimmune*[32] if for every essential $\Sigma^{0,X}_1$ formula $\varphi(U, V)$, $\varphi(R, S)$ holds for some $R \subseteq \overline{A}_0$ and $S \subseteq \overline{A}_1$. ◊

The following exercise shows that dependent hyperimmunity can be seen as a strong form of hyperimmunity. The two notions coincide on co-c.e. sets.

31: The terminology comes from Lerman, Solomon and Towsner [43] who first proved that ADS does not imply CAC over RCA_0. The proof was then simplified by Patey [46].

32: One could as well have defined the notion of dependently constant-bound X-immune by fixing the cardinality of the sets R and S. This would also yield a notion separating ADS from CAC over RCA_0.

Exercise 6.6.6 (Patey [46]). Show that

1. If A_0, A_1 are dependently hyperimmune, then A_0 and A_1 are both hyperimmune.
2. If A_0, A_1 are both hyperimmune and A_0 is co-c.e., then A_0, A_1 are dependently hyperimmune. ⋆

As usual, one can define the corresponding notion of preservation.

Definition 6.6.7. A problem P admits preservation of 1 dependent hyperimmunity if for every set Z and every pair A_0, A_1 of dependently Z-hyperimmune sets, every Z-computable instance X of P admits a solution Y such that A_0, A_1 are dependently $Z \oplus Y$-hyperimmune. ◇

We now prove that ADS admits preservation of 1 dependent hyperimmunity, while we shall see later that CAC does not.

Theorem 6.6.8 (Patey [46])
Let A_0, A_1 be dependently hyperimmune, and $\mathcal{L} = (\mathbb{N}, <_{\mathcal{L}})$ be a computable linear order. Then there is an infinite ascending or descending sequence G such that A_0, A_1 is dependently G-hyperimmune.

Proof. Consider the notion of forcing whose *conditions*[33] are 3-tuples (σ_0, σ_1, X), where

1. (σ_i, X) is a Mathias condition for each $i < 2$;
2. $\sigma_0 \cup \{x\}$ and $\sigma_1 \cup \{x\}$ form respectively an ascending and a descending sequence, for each $x \in X$;
3. X is computable.

33: Note that this notion of forcing for building solutions to ADS is a particular case of the one in Theorem 6.5.7, since any linear order is a degenerate partial order.

A condition (τ_0, τ_1, Y) *extends* (σ_0, σ_1, X) if (τ_i, Y) Mathias extends (σ_i, X) for every $i < 2$. One can therefore see a condition as two simultaneous Mathias conditions sharing a same reservoir. Every filter $\mathcal{F}$ induces two sets: $G_{0,\mathcal{F}}$ and $G_{1,\mathcal{F}}$, defined by $G_{i,\mathcal{F}} = \bigcup\{\sigma_i : (\sigma_0, \sigma_1, X) \in \mathcal{F}\}$.

We make the following hypothesis:

> (H1): For every infinite computable set X, there is some $x_0, x_1 \in X$ such that $\{y \in X : x_0 <_{\mathcal{L}} y\}$ and $\{y \in X : x_1 >_{\mathcal{L}} y\}$ are both infinite.

If the (H1) hypothesis fails for some set X, then one can computably thin it out to obtain a computable infinite ascending or descending sequence $Y \subseteq X$. In particular, A_0, A_1 are dependently Y-hyperimmune, so we are done. We can therefore from now on assume that (H1) holds.

Lemma 6.6.9. Suppose (H1) holds. Let $p = (\sigma_0, \sigma_1, X)$ be a condition and $i < 2$. There is an extension (τ_0, τ_1, Y) of p and some $x > |\sigma_i|$ such that $x \in \tau_i$.⋆

Proof. Say $i = 0$ as the other case is symmetric. By (H1), there is some $x_0 \in X$ such that $Y = \{y \in X : x_0 <_{\mathcal{L}} y\}$ is infinite. Let $\tau_0 = \sigma_0 \cup \{x_0\}$, and $\tau_1 = \sigma_1$. Then, (τ_0, τ_1, Y) is an extension of p such that $x_0 \in \tau_0$. ■

We now define a disjunctive forcing question for Σ^0_1-formulas. Given a condition $p = (\sigma_0, \sigma_1, X)$, a *split pair*[34] is an ordered pair (ρ_0, ρ_1) such that $\rho_i \subseteq X$ for each $i < 2$, ρ_0 is ascending, ρ_1 is descending, and $\max_{\mathcal{L}}(\rho_0) <_{\mathcal{L}} \min_{\mathcal{L}}(\rho_1)$.[35]

34: Note that the notion of split pair is the restriction of split triples from Theorem 6.5.7 to linear orders.

35: In other words, every element of the ascending sequence ρ_0 is below (with respect to $<_{\mathcal{L}}$) every element of the descending sequence ρ_1.

Definition 6.6.10. Let $p = (\sigma_0, \sigma_1, X)$ be a condition and $\varphi_0(G), \varphi_1(G)$ be two Σ^0_1-formulas. Let $p \mathrel{?\vdash} \varphi_0(G_0) \vee \varphi_1(G_1)$ hold if there is a split pair (ρ_0, ρ_1) such that for each $i < 2$, $\varphi_i(\sigma_i \cup \rho_i)$ holds. ◇

Note that being a split pair is a decidable predicate, hence the forcing question is Σ^0_1-preserving. The following lemma shows that the forcing question not only meets its specification, but also satisfies some form of Π^0_1-extremality.

Lemma 6.6.11. Let $p = (\sigma_0, \sigma_1, X)$ be a condition and $\varphi_0(G), \varphi_1(G)$ be two Σ^0_1-formulas.

1. If $p \mathrel{?\vdash} \varphi_0(G_0) \vee \varphi_1(G_1)$, then there is some $i < 2$ and some extension $q \leq p$ forcing $\varphi_i(G_i)$.
2. If $p \mathrel{?\nvdash} \varphi_0(G_0) \vee \varphi_1(G_1)$, then p forces $\neg\varphi_0(G_0) \vee \neg\varphi_1(G_1)$. ★

Proof. Suppose first $p \mathrel{?\vdash} \varphi_0(G_0) \vee \varphi_1(G_1)$ holds, as witnessed by some split pair (ρ_0, ρ_1). By the pigeonhole principle, there is some infinite X-computable subset $Y \subseteq X$ such that for every $x \in \rho_0 \cup \rho_1$, either for every $y \in Y$, $x <_{\mathcal{L}} y$, or for every $y \in Y$, $x >_{\mathcal{L}} y$. We say that x is *small* if it is on the first case and *large* otherwise. If $\max_{\mathcal{L}}(\rho_0)$ is small, then every element in ρ_0 is small, so the condition $(\sigma_0 \cup \rho_0, \sigma_1, Y)$ is an extension of p forcing $\varphi_0(G_0)$. If $\max_{\mathcal{L}}(\rho_0)$ is large, then every element in ρ_1 is large, so $(\sigma_0, \sigma_1 \cup \rho_1, Y)$ is an extension of p forcing $\varphi_1(G_1)$.

Suppose now $p \mathrel{?\nvdash} \varphi_0(G_0) \vee \varphi_1(G_1)$. Suppose for the contradiction that there is an extension $q = (\tau_0, \tau_1, Y)$ of p such that $\varphi_0(\tau_0)$ and $\varphi_1(\tau_1)$ both hold. Then, letting $\rho_0 = \tau_0 \setminus \sigma_0$ and $\rho_1 = \tau_1 \setminus \sigma_1$, the pair (ρ_0, ρ_1) forms a split pair contradicting our hypothesis. Thus, p already forces $\neg\varphi_0(G_0) \vee \neg\varphi_1(G_1)$. ■

We now prove that for every sufficiently generic filter $\mathcal{F}$, there is some $i < 2$ such that A_0, A_1 is dependently $G_{i,\mathcal{F}}$-hyperimmune.

Lemma 6.6.12. Let $p = (\sigma_0, \sigma_1, X)$ be a condition and $\varphi_0(G, U, V), \varphi_1(G, U, V)$ be two Σ^0_1-formulas. There is some $i < 2$ and an extension q of p forcing $\varphi_i(G_i, U, V)$ not to be essential, or $\varphi_i(G_i, U, V)$ to hold for some sets $U \subseteq \overline{A}_0$ and $V \subseteq \overline{A}_1$. ★

Proof. Let $\psi(U, V)$ be the Σ^0_1-formula which holds if there is some $U_0, U_1 \subseteq U$ and some $V_0, V_1 \subseteq V$ such that $p \mathrel{?\vdash} \varphi_0(G_0, U_0, V_0) \vee \varphi_1(G_1, U_1, V_1)$.

If $\psi(U, V)$ is essential, then by dependent hyperimmunity of A_0, A_1, there are some finite sets $U \subseteq \overline{A}_0$ and $V \subseteq \overline{A}_1$ such that $\psi(U, V)$ holds. Let U_0, U_1, V_0, V_1 witness this. By Lemma 6.6.11(1), there is some $i < 2$ and an extension q of p forcing $\varphi_i(G_i, U_i, V_i)$. Since $U_i \subseteq \overline{A}_0$ and $V_i \subseteq \overline{A}_1$, then q is the desired extension.

Suppose now that $\psi(U, V)$ is not essential. Unfolding the definition, there is some $x \in \mathbb{N}$ such that for every finite set $R > x$, there is some $y_R \in \mathbb{N}$ such that for every finite set $S > y_R$, $\psi(R, S)$ does not hold. Suppose for the contradiction that there is a filter $\mathcal{F}$ containing p such that $\varphi_0(G_{0,\mathcal{F}}, U, V)$ and $\varphi_1(G_{1,\mathcal{F}}, U, V)$ are both essential. For each $i < 2$, since $\varphi_i(G_{i,\mathcal{F}}, U, V)$ is essential, there is some $R_i > x$ such that for every $y \in \mathbb{N}$, there is some $S_i > y$ such that $\varphi_i(G_{i,\mathcal{F}}, R_i, S_i)$ holds. Let $R = R_0 \cup R_1$, and for each $i < 2$, let $S_i > y_R$ be such that $\varphi_i(G_{i,\mathcal{F}}, R_i, S_i)$ holds. Let $S = S_0 \cup S_1$. Then p does not force $\neg\varphi_0(G_0, R_0, S_0) \vee \neg\varphi_1(G_1, R_1, S_1)$, so by Lemma 6.6.11(2), $p \mathrel{?\vdash} \varphi_0(G_0, R_0, S_0) \vee \varphi_1(G_1, R_1, S_1)$. Thus, $\psi(R, S)$ holds, with $R > x$ and $S > y_R$, contradiction. ■

We are now ready to prove Theorem 6.6.8. Let $\mathcal{F}$ be a sufficiently generic filter for this notion of forcing. For each $i < 2$, let $G_i = G_{\mathcal{F},i}$. By Lemma 6.6.9, G_i is infinite for every $i < 2$. Moreover, by construction, G_0 is an ascending sequence and G_1 is a descending sequence. Last, by Lemma 6.6.12, there is some $i < 2$ such that A_0, A_1 is dependently G_i-hyperimmune. This completes the proof of Theorem 6.6.8. ■

We leave the abstract preservation theorem as an exercise.

Exercise 6.6.13. Let $(\mathbb{P}, \leq)$ be a notion of forcing with a Π^0_1-extremal, Σ^0_1-preserving forcing question. Show that for every pair A_0, A_1 of dependently hyperimmune sets and every sufficiently generic filter $\mathcal{F}$, A_0, A_1 is dependently $G_{\mathcal{F}}$-hyperimmune. ⋆

We construct a computable partial order witnessing that CAC does not admit preservation of 1 dependent hyperimmunity. This partial order will satisfy some strong structural properties that we now define. Given a partial order $\mathcal{P} = (D, <_{\mathcal{P}})$, we say that $x \in P$ is *small*, *large* or *isolated* if for all but finitely many $y \in D$, $x \leq_P y$, $x \geq_P y$, or $x|_P y$, respectively. We write $S^*(\mathcal{P})$, $L^*(\mathcal{P})$ and $I^*(\mathcal{P})$ for the set of small, large and isolated elements of $\mathcal{P}$, respectively. A partial order is *weakly stable*[36] if every element is either small, large, or isolated, that is, $D = S^*(\mathcal{P}) \cup L^*(\mathcal{P}) \cup I^*(\mathcal{P})$. A partial order is *stable* if every element is small or isolated, or if every element is large or isolated, that is, $D = S^*(\mathcal{P}) \cup I^*(\mathcal{P})$ or $D = L^*(\mathcal{P}) \cup I^*(\mathcal{P})$.

36: Weak stability is arguably the natural notion of stability for CAC, in that a partial order over $\mathbb{N}$ can be seen as a 3-coloring of $[\mathbb{N}]^2$, and this partial order is weakly stable if the corresponding 3-coloring is stable. The stronger notion of stability was first introduced by Hirschfeldt and Shore [23], who proved that ADS is equivalent to the statement "Every infinite partial order admits an infinite sub-domain over which it is weakly stable."

Theorem 6.6.14 (Patey [46])
There exists a computable, stable partial order $\mathcal{P} = (\mathbb{N}, <_{\mathcal{P}})$ such that the pair $I^*(\mathcal{P}), L^*(\mathcal{P})$ is dependently hyperimmune.

Proof. Fix an enumeration $\varphi_0(U, V), \varphi_1(U, V), \dots$ of all Σ^0_1 formulas. The construction of the partial order $<_{\mathcal{P}}$ is done by a finite injury priority argument with a movable marker procedure. We want to satisfy the following scheme of requirements for each e, where $L^* = L^*(\mathcal{P})$ and $I^* = I^*(\mathcal{P})$.[37]

$$\mathcal{R}_e : \varphi_e(U, V) \text{ essential} \rightarrow (\exists R \subseteq_{\mathtt{fin}} L^*)(\exists S \subseteq_{\mathtt{fin}} I^*)\varphi_e(R, S)$$

37: Note that by stability of $\mathcal{P}$, we will have $L^* \sqcup I^* = \mathbb{N}$, thus in the requirement, one must think of I^* as $\overline{L^*}$ and L^* as $\overline{I^*}$.

The requirements are given the usual priority ordering. We proceed by stages, maintaining two sets I^*, L^* which represent the limit of the partial order $<_{\mathcal{P}}$. At stage 0, $I^*_0 = L^*_0 = \emptyset$ and $<_{\mathcal{P}}$ is nowhere defined. Moreover, each requirement $\mathcal{R}_e$ is given a movable marker m_e initialized to 0.

A strategy for $\mathcal{R}_e$ *requires attention at stage* $s+1$ if $\varphi_e(R, S)$ holds for some $R < S \subseteq (m_e, s]$. The strategy sets $I^*_{s+1} = (I^*_s \setminus (m_e, min(S)) \cup [min(S), s]$ and $L^*_{s+1} = (L^*_s \setminus [min(S), s]) \cup (m_e, min(S))$. Note that $R \subseteq (m_e, min(S))$ since $R < S$. Then it is declared *satisfied* and does not act until some strategy of higher priority changes its marker. Each marker $m_{e'}$ of strategies of lower priorities is assigned the value $s + 1$.

At stage $s + 1$, assume that $I^*_s \cup L^*_s = [0, s)$ and that $<_{\mathcal{P}}$ is defined for each pair over $[0, s)$.[38] For each $x \in [0, s)$, set $x <_{\mathcal{P}} s$ if $x \in L^*_s$ and $x|_{\mathcal{P}} s$ if $x \in I^*_s$. If some strategy requires attention at stage $s+1$, take the least one and satisfy it. If no such requirement is found, set $L^*_{s+1} = L^*_s$ and $I^*_{s+1} = I^*_s \cup \{s\}$.[39] Then go to the next stage. This ends the construction.

38: By "$<_{\mathcal{P}}$ is defined over $[0, s)$", we don't mean that it is a linear order on $[0, s)$, but that the status "below/above/incomparable" is defined for every pair over $[0, s)$.

39: This choice is arbitrary. One could have defined $L^*_{s+1} = L^*_s \cup \{s\}$ and $I^*_{s+1} = I^*_s$.

Each time a strategy acts, it changes the markers of strategies of lower priority, and is declared satisfied. Once a strategy is satisfied, only a strategy of higher priority can injure it. Therefore, each strategy acts finitely often and the markers stabilize. It follows that $\lim_s I^*_s$ and $\lim_s L^*_s$ both exist, and that $(\mathbb{N}, <_{\mathscr{P}})$ is stable.

Claim. For every $x < y < z$, if $x <_{\mathscr{P}} y$ and $y <_{\mathscr{P}} z$, then $x <_{\mathscr{P}} z$.

Proof. Suppose that $x <_{\mathscr{P}} y$ and $y <_{\mathscr{P}} z$ but $x |_{\mathscr{P}} z$. By construction of $<_{\mathscr{P}}$, $x \in I^*_z$, $x \in L^*_y$ and $y \in L^*_z$. Let $s \leq z$ be the last stage such that $x \in L^*_s$. Then at stage $s+1$, some strategy $\mathscr{R}_e$ receives attention and moves x to I^*_{s+1} and therefore moves $[x, s]$ to I^*_{s+1}. In particular $y \in I^*_{s+1}$ since $y \in [x, s]$. Moreover, the strategies of lower priority have had their marker moved to $s+1$ and therefore will never move any element below s. Since $y <_{\mathscr{P}} z$, then $y \in L^*_z$. In particular, some strategy $\mathscr{R}_i$ of higher priority moved y to L^*_{t+1} at stage $t+1$ for some $t \in (s, z)$. Since $\mathscr{R}_i$ has a higher priority, $m_i \leq m_e$, and since y is moved to L^*_{t+1}, then so is $[m_i, y]$, and in particular $x \in L^*_{t+1}$ since $m_i \leq m_e \leq x \leq y$. This contradicts the maximality of s. ■

Claim. For every $e \in \omega$, $\mathscr{R}_e$ is satisfied.

Proof. By induction over the priority order. Let s_0 be a stage after which no strategy of higher priority will ever act. By construction, m_e will not change after stage s_0. If $\varphi_e(U, V)$ is essential, then $\varphi_e(R, S)$ holds for two sets $m_e < R < S$. Let $s = 1 + max(s_0, S)$. The strategy $\mathscr{R}_e$ will require attention at some stage before s, will receive attention, be satisfied and never be injured. ■

This last claim finishes the proof of Theorem 6.6.14. ■

Corollary 6.6.15 (Lerman, Solomon and Towsner [43])
ADS *does not imply* CAC *over* RCA_0.

Proof. Let $\mathscr{P} = (\mathbb{N}, <_{\mathscr{P}})$ be the partial order of Theorem 6.6.14, and let $A_0 = I^*(\mathscr{P})$ and $A_1 = L^*(\mathscr{P})$. Let H be either infinite chain, or an infinite antichain, and let $\varphi(U, V)$ be the essential $\Sigma^0_1(H)$-formula "$U \cup V \subseteq H$". If H is a chain, then by stability of $\mathscr{P}$, it is an ascending sequence, hence $H \subseteq A_1$. If H is an antichain, then $H \subseteq A_0$. In both cases, φ witnesses the fact that A_0, A_1 is not dependently H-hyperimmune. Thus CAC does not admit preservation of 1 dependent hyperimmunity. On the other hand, by Theorem 6.6.8, ADS admits preservation of 1 dependent hyperimmunity. Thus, by Corollary 6.1.4, ADS does not imply CAC over RCA_0. ■